ARTICLE IN PRESS
APAL:2573

Annals of Pure and Applied Logic ••• (••••) •••–•••

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Forking in short and tame abstract elementary classes ☆

Will Boney [a,*], Rami Grossberg [b]

[a] *Mathematics Department, Harvard University, Cambridge, MA, USA*
[b] *Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA, USA*

ARTICLE INFO

Article history:
Received 13 March 2015
Received in revised form 3 January 2017
Accepted 4 January 2017
Available online xxxx

MSC:
primary 03C48, 03C45

Keywords:
Abstract Elementary Classes
Tameness
Classification theory

ABSTRACT

We develop a notion of forking for Galois-types in the context of Abstract Elementary Classes (AECs). Under the hypotheses that an AEC K is tame, type-short, and failure of an order-property, we consider

Definition 1. Let $M_0 \prec N$ be models from K and A be a set. We say that the Galois-type of A over N *does not fork over* M_0, written $A \underset{M_0}{\downarrow} N$, iff for all small $a \in A$ and all small $N^- \prec N$, we have that Galois-type of a over N^- is realized in M_0.

Assuming property (E) (Existence and Extension, see Definition 3.3) we show that this non-forking is a well behaved notion of independence, in particular satisfies symmetry and uniqueness and has a corresponding U-rank. We find conditions for a universal local character, in particular derive superstability-like property from little more than categoricity in a "big cardinal". Finally, we show that under large cardinal axioms the proofs are simpler and the non-forking is more powerful.
In [10], it is established that, if this notion is an independence notion, then it is the only one.

1. Introduction

Much of modern model theory has focused on Shelah's forking. In the last twenty years, significant progress has been made towards understanding of unstable theories, especially simple theories (Kim [33] and Kim and Pillay [34]), NIP theories (surveys by Adler [1] and Simon [56]), and, most recently, NTP_2 (Ben-Yaacov and Chernikov [6] and Chernikov, Kaplan, and Shelah [16]).

In the work on classification theory for Abstract Elementary Classes (AECs), such a nicely behaved notion is not known to exist. However, much work has been done towards this goal. Around 2005, homogeneous

☆ Part of this material is based upon work done while the first author was supported by the National Science Foundation under Grant No. DMS-1402191.
* Corresponding author.
E-mail addresses: wboney@math.harvard.edu (W. Boney), rami@cmu.edu (R. Grossberg).

http://dx.doi.org/10.1016/j.apal.2017.02.002
0168-0072/© 2017 Elsevier B.V. All rights reserved.

model theory – working under the assumption that there exists a monster model which is sequential homogeneous (but not necessarily saturated as in the first-order sense) and types consists of sets of first-order formulas – reached a stage of development that parallels that of first-order model theory in the seventies. There is a Morley-like categoricity theorem (Keisler [32] and Lessmann [37]), forking exists (Buechler and Lessmann [14]), and even a main gap is true (Grossberg and Lessmann [20]). Hyttinen and Kesälä studied a further extension of homogeneous model theory called finitary AECs in [27] and in [28]. They established both Morley's categoricity theorem and that non-splitting is a variant of forking under the assumptions of $\aleph_0$-stability and what they call simplicity (like our extension property) in a countable language.

However, as AECs are much more general the situation for AECs is more complicated. There are classes axiomatized by $L_{\omega_1,\omega}$ that do not fit into the framework of homogeneous model theory:

(1) Marcus [41] constructed an $L_{\omega_1,\omega}$ sentence that is categorical in all cardinals but does not have even an $\aleph_1$-homogeneous model.
(2) Hart and Shelah [26] constructed, for each $k < \omega$, an $L_{\omega_1,\omega}$ sentence ψ_k which is categorical in all $\aleph_n$ for $n \leq k$ but not categorical in higher cardinals. By the categoricity theorem for finitary AECs [28], this means that $\text{Mod}(\psi_k)$ is not homogeneous as it is not even finitary.

In [46, Chapter N], Shelah explains the importance of classification theory for AECs. At the referee's suggestion, we summarize the argument here, although the truly interested reader should consult the source.

As mentioned above, classification theory has become the main focus of model theory. Shelah and other early workers were motivated by purely abstract problems, such as the main gap in [43]. The machinery used to solve these problems turned out to be very powerful and, about 20 years later, Chatzidikis, Hrushovski, Scanlon, and others discovered deep applications to geometry, algebra, and other fields.

However, this powerful machinery was restricted because it only applied to first-order model theory. This is natural from a logical point of view as first-order logic has many unique features (e.g., compactness), but there are many mathematical classes that are not first order axiomatizable: we list some in this introduction and in Section 5 and each of Grossberg [17], Baldwin [2], and [46, Chapter N] contain their own lists. The logic needed to axiomatize each context varies, from $L_{\omega_1,\omega}(Q)$ for quasiminimal classes to $L_{|R|^+,\omega}$ for torsion R-modules. Varying the substructure relation (e.g., subgroup vs. pure subgroup) complicates the picture further.

A unifying perspective is given by AECs and Shelah began their classification (and their study) in the late 1970s. Here again questions of number of nonisomorphic models have formed the basic test questions. The most central one here is Shelah's Categoricity Conjecture; Shelah proposed this conjecture for $L_{\omega_1,\omega}$ in the late seventies as a way to measure the development of the relevant classification theory. At present, there are many partial results that approximates this conjecture and harder questions for AECs. Despite an estimated of more than 2,000 published pages, the full conjecture is not within reach of current methods, in contrast to the existence of relatively simple proofs of the conjecture for the cases of homogeneous models and finitary AECs. Due to the lack of compactness and syntax, extra set-theoretic assumptions (in addition to new techniques) have been needed to get these results; the strong Devlin–Shelah diamonds on successors in [51] and large cardinals in [40] are excellent examples of this.

Differing from Shelah, our vision is that model-theoretic assumptions (especially tameness and type-shortness here) will take the place of set-theoretic ones. The hope here is two-fold: first, that, although these assumptions don't hold everywhere, they can be shown to hold in many natural and, second, that these model-theoretic assumptions are enough to develop a robust classification theory. This paper (and follow-ups by Boney, Grossberg, Kolesnikov, and Vasey) provide evidence for the first hope and examples described in Section 5 provide evidence for the first.

In [49], Shelah introduced analogues of splitting and strong splitting for AECs. Building on this, Shelah [51], and Grossberg, VanDieren, and Villaveces [24], Grossberg and VanDieren [21,23] used tameness