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Disjoint Borel functions ☆

Dan Hathaway

Mathematics Department, University of Denver, Denver, CO 80208, USA

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ABSTRACT

For each $a \in {}^\omega\omega$, we define a Baire class one function $f_a : {}^\omega\omega \to {}^\omega\omega$ which encodes a in a certain sense. We show that for each Borel $g : {}^\omega\omega \to {}^\omega\omega$, $f_a \cap g = \emptyset$ implies $a \in \Delta^1_1(c)$ where c is any code for g. We generalize this theorem for g in a larger pointclass Γ. Specifically, when $\Gamma = \boldsymbol{\Delta}^1_2$, $a \in L[c]$. Also for all $n \in \omega$, when $\Gamma = \boldsymbol{\Delta}^1_{3+n}$, $a \in \mathcal{M}_{1+n}(c)$.

1. Introduction

Definition 1.1. A challenge–response relation (*c.r.-relation*) is a triple $\langle R_-, R_+, R\rangle$ such that $R \subseteq R_- \times R_+$. The set R_- is the set of *challenges*, and R_+ is the set of *responses*. When cRr, we say that r *meets* c.

Definition 1.2. A backwards generalized Galois–Tukey connection (*morphism*) from $\mathcal{A} = \langle A_-, A_+, A\rangle$ to $\mathcal{B} = \langle B_-, B_+, B\rangle$ is a pair $\langle \phi_-, \phi_+\rangle$ of functions $\phi_- : B_- \to A_-$ and $\phi_+ : A_+ \to B_+$ such that

$$(\forall c \in B_-)(\forall r \in A_+)\, \phi_-(c)\, A\, r \Rightarrow c\, B\, \phi_+(r).$$

When there is a morphism from $\mathcal{A}$ to $\mathcal{B}$, let us say that $\mathcal{A}$ is *above* $\mathcal{B}$ and $\mathcal{B}$ is *below* $\mathcal{A}$.

Definition 1.3. The *norm* of a c.r.-relation $\mathcal{R} = \langle R_-, R_+, R\rangle$ is

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E-mail address: Daniel.Hathaway@du.edu.

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$$||\mathcal{R}|| := \min\{|S| : S \subseteq R_+ \text{ and } (\forall c \in R_-)(\exists r \in S)\, c\, R\, r\}.$$

If there is a morphism from $\mathcal{A}$ to $\mathcal{B}$, then $||\mathcal{A}|| \geq ||\mathcal{B}||$. Challenge–response relations and morphisms between them were introduced by Vojtas as a way to abstract features of the study of cardinal characteristics of the continuum. For more on c.r.-relations, see [2] and [6].

Temporarily fix a pointclass Γ. Let $\mathcal{F}_\Gamma$ be the set of functions from ${}^\omega\omega$ to ${}^\omega\omega$ in Γ. Let D be the binary relation of disjointness of functions from ${}^\omega\omega$ to ${}^\omega\omega$. That is, given two functions $f, g : {}^\omega\omega \to {}^\omega\omega$, let

$$f\, D\, g :\Leftrightarrow f \cap g = \emptyset \Leftrightarrow (\forall x \in {}^\omega\omega)\, f(x) \neq g(x).$$

Let $\mathcal{D}_\Gamma$ be the c.r.-relation

$$\mathcal{D}_\Gamma := \langle \mathcal{F}_\Gamma, \mathcal{F}_\Gamma, D\rangle.$$

In this paper we will be interested in the c.r.-relation $\mathcal{D}_\Gamma$ for various pointclasses Γ.

For example, we will be interested in computing $||\mathcal{D}_{\mathbf{\Delta}^1_1}||$, which is the smallest size of a family of Borel functions from ${}^\omega\omega$ to ${}^\omega\omega$ such that each Borel function from ${}^\omega\omega$ to ${}^\omega\omega$ is disjoint from some member of the family. We will show that $||\mathcal{D}_{\mathbf{\Delta}^1_1}|| = 2^\omega$ by showing that $\mathcal{D}_{\mathbf{\Delta}^1_1}$ is above a c.r.-relation whose norm is 2^ω. Specifically, we will show that $\mathcal{D}_{\mathbf{\Delta}^1_1}$ is above $\langle {}^\omega\omega, {}^\omega\omega, \leq_{\Delta^1_1}\rangle$, where $a \leq_{\Delta^1_1} b$ iff $a \in {}^\omega\omega$ is definable by a Δ^1_1 formula using $b \in {}^\omega\omega$ as a parameter. To define the ϕ_- part of the morphism, for each $a \in {}^\omega\omega$ we will define a Baire class one function $f_a : {}^\omega\omega \to {}^\omega\omega$ (and we will have $\phi_-(a) = f_a$). The ϕ_+ part of the morphism will simply map each function from ${}^\omega\omega$ to ${}^\omega\omega$ in Γ to any code for that function. The fact that $\langle \phi_-, \phi_+\rangle$ is a morphism is the following statement: for each $a \in {}^\omega\omega$ and Borel function $g : {}^\omega\omega \to {}^\omega\omega$,

$$f_a \cap g = \emptyset \Rightarrow a \leq_{\Delta^1_1} \text{any code for } g.$$

We will prove that there is a morphism from $\mathcal{D}_{\mathbf{\Delta}^1_1}$ to $\langle {}^\omega\omega, {}^\omega\omega, \leq_{\Delta^1_1}\rangle$ by proving a general theorem (Theorem 5.3) which provides a sufficient condition for when there exists a morphism from an arbitrary $\mathcal{D}_\Gamma$ to an arbitrary $\langle {}^\omega\omega, {}^\omega\omega, \prec\rangle$, where $\prec$ is an ordering on ${}^\omega\omega$. Just like the case with $\mathcal{D}_{\mathbf{\Delta}^1_1}$, we will use the functions f_a for the ϕ_- map, and the ϕ_+ map will be "take any code for". Thus, if the appropriate relationship holds between Γ and $\prec$, then we will have that for each $a \in {}^\omega\omega$ and each $g : {}^\omega\omega \to {}^\omega\omega$ in Γ,

$$f_a \cap g = \emptyset \Rightarrow a \prec \text{any code for } g.$$

We will get that there exists a morphism from $\mathcal{D}_{\mathbf{\Delta}^1_2}$ to $\langle {}^\omega\omega, {}^\omega\omega, \leq_L\rangle$, where $a \leq_L b$ iff $a \in L[b]$. The analogous result for larger Γ uses large cardinals. We will have that as long as $\mathcal{M}_1(b)$ (the canonical inner model containing 1 Woodin cardinal and containing $b \in {}^\omega\omega$) exists for all $b \in {}^\omega\omega$, then there is a morphism from $\mathcal{D}_{\mathbf{\Delta}^1_3}$ to $\langle {}^\omega\omega, {}^\omega\omega, \leq_{\mathcal{M}_1}\rangle$, where $a \leq_{\mathcal{M}_1} b$ iff $a \in \mathcal{M}_1(b)$. Next, as long as $\mathcal{M}_2(b)$ exists for all $b \in {}^\omega\omega$, there is a morphism from $\mathcal{D}_{\mathbf{\Delta}^1_4}$ to $\langle {}^\omega\omega, {}^\omega\omega, \leq_{\mathcal{M}_2}\rangle$. The pattern continues like this through the projective hierarchy.

In this paper, we are considering functions from ${}^\omega\omega$ to ${}^\omega\omega$ in a pointclass Γ. We could have instead considered functions in Γ from an arbitrary uncountable Polish space X to an arbitrary Polish space Y, and our results would not have changed much. The appropriate encoding function $f''_a : X \to Y$ could be defined by first defining $f'_a : {}^\omega 2 \to {}^\omega\omega$ in a way similar to f_a and then using an injection of ${}^\omega 2$ into X and a surjection of ${}^\omega\omega$ onto Y. We trust that the interested reader can work through the details without trouble.

2. Related results

Before considering $\mathcal{D}_\Gamma$ for various Γ, we will consider related c.r.-relations. First, consider the everywhere domination ordering of functions from ${}^\omega\omega$ to ω. That is, given $f, g : {}^\omega\omega \to \omega$, we write $f \leq g$ iff