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Shelah's eventual categoricity conjecture in universal classes: Part I ☆

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ABSTRACT

We prove:

Theorem 0.1. *Let K be a universal class. If K is categorical in cardinals of arbitrarily high cofinality, then K is categorical on a tail of cardinals.*

The proof stems from ideas of Adi Jarden and Will Boney, and also relies on a deep result of Shelah. As opposed to previous works, the argument is in ZFC and does not use the assumption of categoricity in a successor cardinal. The argument generalizes to abstract elementary classes (AECs) that satisfy a locality property and where certain prime models exist. Moreover assuming amalgamation we can give an explicit bound on the Hanf number and get rid of the cofinality restrictions:

Theorem 0.2. *Let K be an AEC with amalgamation. Assume that K is fully $LS(K)$-tame and short and has primes over sets of the form $M \cup \{a\}$. Write $H_2 := \beth_{\left(2^{\beth_{(2^{LS(K)})^+}}\right)^+}$. If K is categorical in a $\lambda > H_2$, then K is categorical in all $\lambda' \geq H_2$.*

1. Introduction

Morley's categoricity theorem [39] states that a first-order countable theory that is categorical in some uncountable cardinal must be categorical in all uncountable cardinals. The result motivated much of the development of first-order classification theory (it was later generalized by Shelah [43] to uncountable theories).

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Toward developing a classification theory for non-elementary classes, one can ask whether there is such a result for infinitary logics, e.g. for an $L_{\omega_1,\omega}$ sentence. In 1971, Keisler proved [30, Section 23] a generalization of Morley's theorem to this framework assuming in addition that the model in the categoricity cardinal is sequentially homogeneous. Unfortunately Shelah later observed using an example of Marcus [38] that Keisler's assumption does not follow from categoricity. Still in the late seventies Shelah proposed the following far-reaching conjecture:

Conjecture 1.1 *(Open problem D.(3a) in [49]). If L is a countable language and $\psi \in L_{\omega_1,\omega}$ is categorical in one $\lambda \geq \beth_{\omega_1}$, then it is categorical in all $\lambda' \geq \beth_{\omega_1}$.*

This has now become the central test problem in classification theory for non-elementary classes. Shelah alone has more than 2000 pages of approximations (for example [44–46,37,50,52–54]). Shelah's results led him to introduce a semantic framework encompassing several different infinitary logics and algebraic classes [47]: abstract elementary classes (AECs). In this framework, we can state an eventual version of the conjecture[1]:

Conjecture 1.2 *(Shelah's eventual categoricity conjecture for AECs). An AEC that is categorical in a high-enough cardinal is categorical on a tail of cardinals.*

Remark 1.3. A more precise statement is that there should be a function $\mu \mapsto \lambda_\mu$ such that every AEC K categorical in some $\lambda \geq \lambda_{\mathrm{LS}(K)}$ is categorical in all $\lambda' \geq \lambda_{\mathrm{LS}(K)}$. By a similar argument as for the existence of Hanf numbers [22] (see [1, Conclusion 15.13]), Shelah's eventual categoricity conjecture for AECs is equivalent to the statement that an AEC categorical in *unboundedly many* cardinals is categorical on a tail of cardinals. We will use this equivalence freely. Note that Theorem 0.2 gives an *explicit*[2] bound for λ_μ, so proves a stronger statement than just Shelah's eventual categoricity conjecture for universal classes with amalgamation.[3]

Positive results are known in less general frameworks: For homogeneous model theory by Lessmann [34] and more generally for $\aleph_0$-tame[4] simple finitary AECs by Hyttinen and Kesälä [25] (note that these results apply only to countable languages). In uncountable languages, Grossberg and VanDieren proved the conjecture in tame AECs categorical in a successor cardinal [20,18]. Later Will Boney pointed out that tameness follows[5] from large cardinals [7], a result that (as pointed out in [35]) can also be derived from a 25 year old theorem of Makkai and Paré ([36, Theorem 5.5.1]). A combination of this gives that statements much stronger than Shelah's categoricity conjecture for a successor hold if there exists a proper class of strongly compact cardinals.

The question of whether categoricity in a sufficiently high *limit* cardinal implies categoricity on a tail remains open (even in tame AECs). The central tool there is the notion of a *good λ-frame*, a local axiomatization of forking which is the main concept in [53]. After developing the theory of good λ-frames over several hundreds of pages, Shelah claims to be able to prove the following (see [53, Discussion III.12.40], a proof should appear in [42]):

[1] The statement here appears in [53, Conjecture N.4.2].
[2] We thank John Baldwin for helpful conversation on the topic.
[3] We are not sure how to make the distinction precise. Maybe one can call the *computable* eventual categoricity conjecture the statement that has the additional requirement that $\mu \mapsto \lambda_\mu$ be computable, where computable can be defined as in [5]. Note that in Shelah's original categoricity conjecture, λ_μ is $\beth_{(2^\mu)^+}$, see [51, 6.14.(3)].
[4] Tameness is a locality property for orbital types introduced by Grossberg and VanDieren in [19].
[5] Recently Boney and Unger [10] established that the statement "all AECs are tame" is in fact *equivalent* to a large cardinal axioms (the existence of a proper class of almost strongly compact cardinals). This result does not however say anything on the consistency strength of Shelah's eventual categoricity conjecture.