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The strength of infinitary Ramseyan principles can be accessed by their densities ☆

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ABSTRACT

In this article, we conduct a model-theoretic investigation of three infinitary Ramseyan statements: the Infinite Ramsey Theorem for pairs and two colours (RT^2_2), the Canonical Ramsey Theorem for pairs (CRT^2) and the Regressive Ramsey Theorem for pairs (RegRT^2). We approximate the logical strength of these principles by the strength of their first-order iterated versions, known as density principles. We then investigate their logical strength using strong initial segments of models of Peano Arithmetic, in the spirit of the classical article by Paris and Kirby, hereby re-proving old results model-theoretically. The article is concluded by a discussion of two further outreaches of densities. One is a further investigation of the strength of the Ramsey Theorem for pairs. The other deals with the asymptotics of the standard Ramsey function R^2_2.

This article stands at the cross-roads of three subjects: unprovability theory (metamathematics), Ramsey theory and model theory of arithmetic. Our goals and motivation come from unprovability theory, the statements whose strength we study are Ramsey-theoretic, and the methods of obtaining our results are model-theoretic. The paper has two purposes. The first purpose is to learn to extract Π^0_2 fragments from Π^1_2 combinatorial principles that possess strength. The second purpose is to develop the model theory of some Ramseyan principles in the spirit of Paris and Kirby, which is partly a pedagogical task. In this second capacity, Sections 4, 5 and 6 below can be considered as accompanying the article [1], and can be a good

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starting point for learning model-theoretic methods in metamathematics, before moving on to study modern developments, for example the article [2].

The story of the study of the strength of second-order arithmetical statements started with the emergence of reverse mathematics in mid-1970s. Among the very first developments was the seminal article [8] of L. Kirby and J. Paris, where they, among other things, proved that for every $k \geq 3$, the Infinite Ramsey Theorem for dimension k has exactly the same first-order arithmetical consequences (over a weak base theory) as Peano Arithmetic. The method of proof was model-theoretic, using semi-regular and strong initial segments of models of arithmetic. In the class of semi-regular initial segments I of a model of arithmetic whose second-order part is defined as the collection of all intersections of M-definable sets with I, I satisfies $I \to (I)^3_2$ if and only if I is strong, and thus satisfies full Peano Arithmetic. The method of proof is closely related to the subsequent method of indicators and the discovery of PA-unprovable combinatorial statements.

When the dimension is not fixed, the logical strength is higher than PA: $\mathrm{ACA}_0 + \mathrm{IRT}$ can be axiomatized through the theory ACA_0' (that is the theory ACA_0 + $\{\forall X \forall n$, "the n-th Turing jump of X exists"$\}$, see [12] and [9]). In conjunction with other results, it follows that the proof-theoretic ordinal of the Infinite Ramsey Theorem is ε_ω. The crucial notion used in the proof of this result is that of *density*, which allowed the formulation of an iterated Paris–Harrington Principle $\mathrm{PH}^{(n)}$ and the proof that the Π^0_2-consequences of the Infinite Ramsey Theorem are the same as the Π^0_2-consequences of the first-order theory $\mathrm{PA} + \bigcup_{n\in\omega} \mathrm{PH}^{(n)}$. Around the same time, in the early 1980s, another version of *density* was used to establish the strength of the Open Ramsey Theorem [5].

Throughout this article, we shall be interested in another, weaker notion of density, which is the original notion due to Jeff Paris [11].

Definition 1. We say that a set X is 0-dense(c, k) if $|X| > \min X + 1$. For every $n \in \omega$, X is $(n+1)$-dense(c, k) if for every $f \colon [X]^k \to c$, there is an f-homogeneous n-dense(c, k) subset of X.

The statement "for all a, n, there exists b such that $[a, b]$ is an n-dense$(2, 3)$ set" was the first combinatorial PA-unprovable statement [11]. We shall be interested in statements of the form "for all a there is b such that $[a, b]$ is n-dense$(2, 2)$" and variations of these notions for regressive colourings and for canonical sets. Using these notions, we shall obtain approximations of infinitary principles by first-order theories. For Ramsey Theorem for pairs and two colours (RT^2_2), Canonical Ramsey Theorem for pairs (CRT^2) and Regressive Ramsey Theorem for pairs (RegRT^2), we show that the corresponding infinitary principle is Π^0_2-conservative over the corresponding theory of the form $I\Sigma_1 + \bigcup_{n\in\omega}$ "for all a there is b such that $[a, b]$ is an n-dense set".

This work was motivated, in particular, by the long-standing open problem on determining the strength of RT^2_2, which was recently resolved by Patey and Yokoyama. In principle, the characterisation via densities allows for a purely combinatorial approach to determine the set of all provably recursive functions of RT^2_2, which has been one of the major open problems in reverse mathematics (see problem 13.2 in [3]).

If it were possible to show that for each n there exists a primitive recursive function $p_n \colon \mathbb{N} \to \mathbb{N}$ such that for all $a \in \mathbb{N}$, the interval $[a, p_n(a)]$ is n-dense$(2, 2)$, then we could conclude that the theory $\mathrm{WKL}_0 + \mathrm{RT}^2_2$ does not prove the totality of the Ackermann function. On the other hand, if it were possible to find $n \in \mathbb{N}$ such that for all but finitely many $a \in \mathbb{N}$, the interval $[a, A(a, a)]$ is not n-dense$(2, 2)$, then $\mathrm{WKL}_0 + \mathrm{RT}^2_2$ would prove the totality of the Ackermann function. So, either way, the problem of the strength of RT^2_2 would be solved.

We note in passing that the following principle MIS, which is an immediate corollary of RT^2_2, is already strong enough to generate all primitive recursive functions over $\mathrm{WKL}^\star_0$ (see, for example, [12] for the definition). MIS stands for: "every sequence $F \colon \mathbb{N} \to \mathbb{N}$ has a weakly monotone increasing subsequence".

The statement RT^2_2 similarly yields the principle CAC and an infinitary tournament principle EM, whose finitary version was investigated by Erdös and Moser. Both principles are defined in Section 7 below. We