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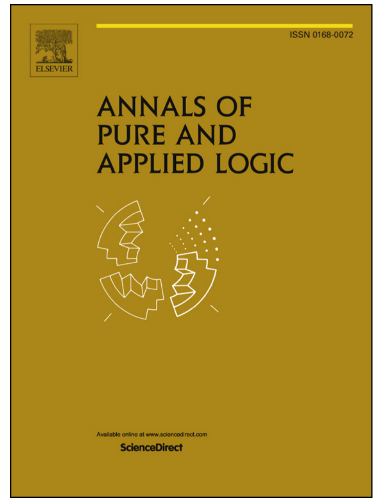

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Spector Bar Recursion over Finite Partial Functions

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Abstract

We introduce a new, demand-driven variant of Spector's bar recursion in the spirit of the Berardi-Bezem-Coquand functional of [4]. The recursion takes place over finite partial functions u, where the control parameter ω, used in Spector's bar recursion to terminate the computation at sequences s satisfying $\omega(\hat{s}) < |s|$, now acts as a guide for deciding exactly where to make bar recursive updates, terminating the computation whenever $\omega(\hat{u}) \in \mathrm{dom}(u)$. We begin by exploring theoretical aspects of this new form of recursion, then in the main part of the paper we show that demand-driven bar recursion can be directly used to give an alternative functional interpretation of classical countable choice. We provide a short case study as an illustration, in which we extract a new bar recursive program from the proof that there is no injection from $\mathbb{N} \to \mathbb{N}$ to $\mathbb{N}$, and compare this with the program that would be obtained using Spector's original variant. We conclude by formally establishing that our new bar recursor is primitive recursively equivalent to the original Spector bar recursion, and thus defines the same class of functionals when added to Gödel's system T.

Keywords. Spector's bar recursion, Dialectica interpretation, program extraction, countable choice.

MSC 2010. 03D65, 03F03, 03F10, 03F25.

1 Introduction

In 1962 C. Spector extended Gödel's functional, or *Dialectica*, interpretation of classical arithmetic [13] to full classical analysis by proving that the functional interpretation of the negative translation of countable choice, and hence full arithmetical comprehension, could be realized by a novel form of recursion which has come to be known as *Spector's bar recursion* [24]. Since then, this seminal work has been extended in several ways, and in particular a number of novel variants of bar recursion have been devised to give computational interpretations to classical analysis in new settings, to the extent that bar recursion, in one form or another, is one of the most recognisable methods of giving a computational interpretation to mathematical analysis.

Spector's original aim was to extend Gödel's proof of the relative consistency of Peano arithmetic to classical analysis. For this purpose, bar recursion is very well suited, allowing us to elegantly and easily expand the soundness of the Dialectica interpretation to incorporate the double negation shift and thus classical countable choice. However, in recent decades applications of proof interpretations such as the Dialectica interpretation and modified realizability have moved away from foundational concerns and towards the more practical issue of extracting computational content from proofs. In line with this shift of emphasis comes an increasing interest in how the modes of computation assigned to non-constructive principles behave.

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