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Spatial logic of tangled closure operators and modal mu-calculus

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ABSTRACT

There has been renewed interest in recent years in McKinsey and Tarski's interpretation of modal logic in topological spaces and their proof that S4 is the logic of any separable dense-in-itself metric space. Here we extend this work to the modal mu-calculus and to a logic of tangled closure operators that was developed by Fernández-Duque after these two languages had been shown by Dawar and Otto to have the same expressive power over finite transitive Kripke models. We prove that this equivalence remains true over topological spaces.

We extend the McKinsey–Tarski topological 'dissection lemma'. We also take advantage of the fact (proved by us elsewhere) that various tangled closure logics with and without the universal modality $\forall$ have the finite model property in Kripke semantics. These results are used to construct a representation map (also called a d-p-morphism) from any dense-in-itself metric space X onto any finite connected locally connected serial transitive Kripke frame.

This yields completeness theorems over X for a number of languages: (i) the modal mu-calculus with the closure operator $\Diamond$; (ii) $\Diamond$ and the tangled closure operators $\langle t\rangle$ (in fact $\langle t\rangle$ can express $\Diamond$); (iii) $\Diamond, \forall$; (iv) $\Diamond, \forall, \langle t\rangle$; (v) the derivative operator $\langle d\rangle$; (vi) $\langle d\rangle$ and the associated tangled closure operators $\langle dt\rangle$; (vii) $\langle d\rangle, \forall$; (viii) $\langle d\rangle, \forall, \langle dt\rangle$. Soundness also holds, if: (a) for languages with $\forall$, X is connected; (b) for languages with $\langle d\rangle$, X validates the well-known axiom G_1. For countable languages without $\forall$, we prove strong completeness. We also show that in the presence of $\forall$, strong completeness fails if X is compact and locally connected.

1. Introduction

Modal logic can be given semantics over topological spaces. In this setting, the modality $\Diamond$ can be interpreted in more than one way. The first and most obvious way is as *closure*. Writing $[\![\varphi]\!]$ for the set of points (in a topological model) at which a formula φ is true, $[\![\Diamond\varphi]\!]$ is defined to be the *closure* of $[\![\varphi]\!]$, so that $\Diamond\varphi$ holds at a point x if and only if every open neighbourhood of x contains a point y satisfying φ.

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Then, $\Box$ becomes the interior operator: $[\![\Box\varphi]\!]$ is the interior of $[\![\varphi]\!]$. Early studies of this semantics include [40,41,26–29].

In a seminal result, McKinsey and Tarski [27] proved that the logic of any given separable[1] dense-in-itself metric space in this semantics is S4: it can be axiomatised by the basic modal Hilbert system K augmented by the two axioms $\Box\varphi \to \varphi$ (T) and $\Box\varphi \to \Box\Box\varphi$ (4).

Motivated perhaps by the current wide interest in spatial logic, a wish to present simpler proofs in 'modern language', growing awareness of the work of particular groups such as Esakia's and Shehtman's, or involvement in new settings such as dynamic topology, interest in McKinsey and Tarski's result has revived in recent years. A number of new proofs of it have appeared, some for specific spaces or embodying other variants [30,4,1,31,39,24,17]. Very recently, strong completeness (every countably infinite S4-consistent set of modal formulas is satisfiable in every dense-in-itself metric space) was established by Kremer [20].

In this paper, we seek to extend McKinsey and Tarski's theorem to more powerful languages. We will extend the modal syntax in two separate ways: first, to the mu-calculus, which adds least and greatest fixed points to the basic modal language, and second, by adding an infinite sequence of new modalities $\Diamond_n$ of arity n ($n \geq 1$) introduced in the context of Kripke semantics by Dawar and Otto [7]. The semantics of $\Diamond_n$ is given by the mu-calculus formula

$$\Diamond_n(\varphi_1, \dots, \varphi_n) \equiv \nu q \bigwedge_{1 \leq i \leq n} \Diamond(\varphi_i \wedge q),$$

for a new atom q not occurring in $\varphi_1, \dots, \varphi_n$. The order and multiplicity of arguments of $\Diamond_n$ is immaterial, so we will abbreviate $\Diamond_n(\gamma_1, \dots, \gamma_n)$ to $\langle t\rangle\{\gamma_1, \dots, \gamma_n\}$. Fernández-Duque used this to give the modalities topological semantics, dubbed them *tangled closure modalities* (this is why we use the notation $\langle t\rangle$), and studied them in [9–12].

Dawar and Otto [7] showed that, somewhat surprisingly, the mu-calculus and the tangled modalities have exactly the same expressive power over finite Kripke models with transitive frames. We will prove that this remains true over topological spaces. So the tangled closure modalities offer a viable alternative to the mu-calculus in both these settings.

We go on to determine the logic of an arbitrary dense-in-itself metric space X in these languages. We will show that in the mu-calculus, the logic of X is axiomatised by a system called S4μ comprising Kozen's basic system for the mu-calculus augmented by the S4 axioms, and the tangled logic of X is axiomatised by a system called S4t similar to one in [10]. We will establish strong completeness for countable sets of formulas.

We will also consider the extension of the tangled language with the *universal modality,* '$\forall$'. (Earlier work on the universal modality in topological spaces includes [36,25].) This language can express connectedness: there is a formula C valid in precisely the connected spaces. Adding this and some standard machinery for $\forall$ to the system S4t gives a system called 'S4t.UC'. We will show that every S4t.UC-consistent formula is satisfiable in every dense-in-itself metric space. Thus, the logic of an arbitrary connected dense-in-itself metric space is S4t.UC. We also show that strong completeness fails in general, even for the modal language plus the universal modality.

A second and more powerful spatial interpretation of $\Diamond$ is as the *derivative operator.* Following tradition, when considering this interpretation we will generally write the modal box and diamond as $[d]$ and $\langle d\rangle$. In this interpretation, $[\![\langle d\rangle\varphi]\!]$ is defined to be the set of *strict limit points* of $[\![\varphi]\!]$: so $\langle d\rangle\varphi$ holds at a point x precisely when every open neighbourhood of x contains a point $y \neq x$ satisfying φ. The original closure diamond is expressible by the derivative operator: $\Diamond\varphi$ is equivalent in any topological model to $\varphi \vee \langle d\rangle\varphi$, and $\Box\varphi$ to $\varphi \wedge [d]\varphi$. So in passing to $\langle d\rangle$, we have not reduced the power of the language.

1 The separability assumption was removed in [32].