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Characterizing large cardinals in terms of layered posets ☆

Sean Cox [a], Philipp Lücke [b,*]

[a] *Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, 1015 Floyd Avenue, Richmond, VA 23284, USA*
[b] *Mathematisches Institut, Rheinische Friedrich-Wilhelms-Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany*

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ABSTRACT

Given an uncountable regular cardinal κ, a partial order is κ-stationarily layered if the collection of regular suborders of $\mathbb{P}$ of cardinality less than κ is stationary in $\mathcal{P}_\kappa(\mathbb{P})$. We show that weak compactness can be characterized by this property of partial orders by proving that an uncountable regular cardinal κ is weakly compact if and only if every partial order satisfying the κ-chain condition is κ-stationarily layered. We prove a similar result for strongly inaccessible cardinals. Moreover, we show that the statement that all κ-Knaster partial orders are κ-stationarily layered implies that κ is a Mahlo cardinal and every stationary subset of κ reflects. This shows that this statement characterizes weak compactness in canonical inner models. In contrast, we show that it is also consistent that this statement holds at a non-weakly compact cardinal.

1. Introduction

Since the results presented in this paper are motivated by classical questions on the *productivity of chain conditions* in partial orders, we start with a short introduction to this topic (longer introduction can be found in [15] and [16]). Given an uncountable regular cardinal κ, we let $\mathcal{C}_\kappa$ denote the statement that the product of two partial orders satisfying the κ-chain condition again satisfies the κ-chain condition. Then $\mathcal{C}_\kappa$ implies the non-existence of κ-Souslin trees and $\mathrm{MA}_{\aleph_1}$ implies $\mathcal{C}_{\aleph_1}$. In particular, the statement $\mathcal{C}_{\aleph_1}$ is independent from the axioms of ZFC. A folklore argument shows that $\mathcal{C}_\kappa$ holds if κ is weakly compact. A small modification of this argument yields the statement of the following proposition. Recall that, given

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* Corresponding author.
E-mail addresses: scox9@vcu.edu (S. Cox), pluecke@uni-bonn.de (P. Lücke).

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an uncountable regular cardinal κ, a partial order is κ-Knaster if every κ-sized collection of conditions can be refined to a κ-sized set of pairwise compatible conditions. Given an infinite cardinal ν and a sequence $\langle \mathbb{P}_\gamma \mid \gamma < \lambda \rangle$, the corresponding *$\nu$-support product* consists of all elements $\vec{p}$ of the full product of these partial orders such that the set $\operatorname{supp}(\vec{p}) = \left\{ \gamma < \lambda \mid \vec{p}(\gamma) \neq \mathbb{1}_{\mathbb{P}_\gamma} \right\}$ has cardinality at most ν.

Proposition 1.1. *If κ is a weakly compact cardinal and $\nu < \kappa$, then ν-support products of partial orders satisfying the κ-chain condition are κ-Knaster.*

Proof. Fix a sequence $\langle \mathbb{P}_\gamma \mid \gamma < \lambda \rangle$ of partial orders satisfying the κ-chain condition and a sequence $\langle \vec{p}_\alpha \mid \alpha < \kappa \rangle$ of conditions in the corresponding ν-support product $\vec{\mathbb{P}} = \prod_{\gamma<\lambda} \mathbb{P}_\gamma$. With the help of the Δ-system lemma, we can find $D \in [\kappa]^\kappa$ and a function $r : \nu \longrightarrow \lambda$ such that the set $\{\operatorname{supp}(\vec{p}_\alpha) \mid \alpha \in D\}$ is a Δ-system with root $\operatorname{ran}(r)$. Given $\alpha, \beta \in D$ with $\alpha < \beta$, there is a minimal $c(\alpha, \beta) \leq \nu$ with the property that either $c(\alpha, \beta) < \nu$ and the conditions $\vec{p}_\alpha(r(c(\alpha, \beta)))$ and $\vec{p}_\beta(r(c(\alpha, \beta)))$ are incompatible in $\mathbb{P}_{r(c(\alpha,\beta))}$, or $c(\alpha, \beta) = \nu$ and the conditions $\vec{p}_\alpha$ and $\vec{p}_\beta$ are compatible in $\vec{\mathbb{P}}$. By the weak compactness of κ, there is $H \in [D]^\kappa$ that is homogeneous for the resulting coloring $c : [D]^2 \longrightarrow \nu + 1$. Since each $\mathbb{P}_\xi$ satisfies the κ-chain condition, we can conclude that c is constant on $[H]^2$ with value ν and this shows that the resulting sequence $\langle \vec{p}_\alpha \mid \alpha \in H \rangle$ consists of pairwise compatible conditions in $\vec{\mathbb{P}}$. $\square$

The κ-Knaster property clearly implies the κ-chain condition. This property is typically used because of its nice product behavior: the product of two κ-Knaster partial orders is κ-Knaster, and the product of a κ-Knaster partial order with a partial order satisfying the κ-chain condition satisfies the κ-chain condition. For reasons described next, we are interested in finding alternative proofs of the above proposition.

A series of deep results shows that, for regular cardinals $\kappa > \aleph_1$, many consequences of weak compactness can be derived from the assumption $\mathcal{C}_\kappa$. In [19], Shelah showed that $\mathcal{C}_\kappa$ fails if κ is the successor of a singular cardinal. Rinot showed in [17] that $\mathcal{C}_\kappa$ implies that every stationary subset of κ reflects. In particular, this result can be used to reprove a series of results of Shelah showing that $\mathcal{C}_\kappa$ fails for all successors of uncountable regular cardinals. In addition, Rinot showed in [16] that for $\kappa > \aleph_1$, the principle $\mathcal{C}_\kappa$ implies a failure of $\square(\kappa)$ and therefore it implies that κ is weakly compact in Gödel's constructible universe L. These results suggest an affirmative answer to the following question of Todorčević.

Question 1.2 *(Todorčević, [22, Question 8.4.27]). Are the following statements equivalent for every regular cardinal $\kappa > \aleph_1$?*

(i) *κ is weakly compact.*
(ii) *$\mathcal{C}_\kappa$ holds.*

In this paper, we want to consider properties of partial orders that imply the κ-chain condition, are preserved by forming products and are equivalent to the κ-chain condition if κ is a weakly compact cardinal. Note that such properties provide alternative proofs of Proposition 1.1. It is interesting to consider the question whether the κ-chain condition can be equivalent to such a property at non-weakly compact cardinals, because both possible answers yield interesting statements: a positive answer to this question would answer Todorčević's question in the negative; while a negative answer leads to new characterizations of weak compactness using chain conditions.

We will now introduce the properties of partial orders studied in this paper. Remember that, given a partial order $\mathbb{P}$, we say that $\mathbb{Q} \subseteq \mathbb{P}$ is a regular suborder if the inclusion map preserves incompatibility and maximal antichains in $\mathbb{Q}$ are maximal in $\mathbb{P}$.

Definition 1.3. Given a cardinal κ and a partial order $\mathbb{P}$, we let $\operatorname{Reg}_\kappa(\mathbb{P})$ denote the collection of all regular suborders of $\mathbb{P}$ of cardinality less than κ.

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