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The uniform content of partial and linear orders ☆

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The principle ADS asserts that every linear order on ω has an infinite ascending or descending sequence. This has been studied extensively in the reverse mathematics literature, beginning with the work of Hirschfeldt and Shore [16]. We introduce the principle ADC, which asserts that every such linear order has an infinite ascending or descending chain. The two are easily seen to be equivalent over the base system RCA_0 of second order arithmetic; they are even computably equivalent. However, we prove that ADC is strictly weaker than ADS under Weihrauch (uniform) reducibility. In fact, we show that even the principle SADS, which is the restriction of ADS to linear orders of type $\omega + \omega^*$, is not Weihrauch reducible to ADC. In this connection, we define a more natural stable form of ADS that we call General-SADS, which is the restriction of ADS to linear orders of type $k + \omega$, $\omega + \omega^*$, or $\omega + k$, where k is a finite number. We define General-SADC analogously. We prove that General-SADC is not Weihrauch reducible to SADS, and so in particular, each of SADS and SADC is strictly weaker under Weihrauch reducibility than its general version. Finally, we turn to the principle CAC, which asserts that every partial order on ω has an infinite chain or antichain. This has two previously studied stable variants, SCAC and WSCAC, which were introduced by Hirschfeldt and Jockusch [16], and by Jockusch, Kastermans, Lempp, Lerman, and Solomon [19], respectively, and which are known to be equivalent over RCA_0. Here, we show that SCAC is strictly weaker than WSCAC under even computable reducibility.

1. Introduction

In the quest to understand the logic strength of Ramsey's theorem for pairs, initiated by Jockusch [18], a myriad of related combinatorial principles were introduced and studied in their own right, giving rise to what is now called the reverse mathematics zoo [1]. Two early examples, introduced by Hirschfeldt and Shore [16], were the ascending/descending sequence principle (ADS) and the chain/antichain principle (CAC). ADS

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asserts that every linear order (on ω) has an infinite ascending or descending sequence, while CAC asserts that every partial order (on ω) has an infinite chain or antichain. (See Section 2 for formal definitions.) While these principles have thus far been analyzed from the point of view of reverse mathematics, in this article we study them using the more nuanced framework of Weihrauch reducibility, which we describe below. We refer the reader to Soare [24] and Simpson [23] for general background on computability and reverse mathematics, respectively, and to Hirschfeldt [15, Sections 6 and 9] for a comprehensive survey of reverse mathematical results about Ramsey's theorem and other combinatorial problems.

As is well-known, there is a natural interplay between computability theory and reverse mathematics, with each of the benchmark subsystems of second-order arithmetic broadly corresponding to a particular level of computability-theoretic complexity (see, e.g., [16, Section 1] for details). In fact, this connection is deeper. The majority of principles one considers in reverse mathematics, like Ramsey's theorem, have the syntactic form

$$\forall X\,(\Phi(X) \to \exists Y\,\Psi(X,Y)),$$

where Φ and Ψ are arithmetical predicates. It is common to call such a principle a *problem*, and to call each X such that $\Phi(X)$ holds an *instance* of this problem, and each Y such that $\Psi(X,Y)$ holds a *solution* to X. The instances of RT^n_k are thus the colorings $c : [\omega]^n \to k$, and the solutions to any such c are the infinite homogeneous sets for this coloring. Over RCA_0, an implication between problems (say $\mathsf{Q} \to \mathsf{P}$) can in principle make multiple applications of the antecedent (Q), or split into cases in a non-uniform way; however, in practice, most implications have a simpler shape. To discuss these, we use the following notions of reduction between problems:

Definition 1.1. Let P and Q be problems.

(1) P is *computably reducible* to Q, written $\mathsf{P} \leq_{\mathrm{c}} \mathsf{Q}$, if every instance X of P computes an instance $\widehat{X}$ of Q, such that if $\widehat{Y}$ is any solution to $\widehat{X}$ then there is a solution Y to X computable from $X \oplus \widehat{Y}$.
(2) P is *strongly computably reducible* to Q, written $\mathsf{P} \leq_{\mathrm{sc}} \mathsf{Q}$, if every instance X of P computes an instance $\widehat{X}$ of Q, such that if $\widehat{Y}$ is any solution to $\widehat{X}$ then there is a solution Y to X computable from $\widehat{Y}$.
(3) P is *Weihrauch reducible* to Q, written $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$, if there are Turing functionals Φ and Δ such that if X is any instance of P then Φ^X is an instance of Q, and if $\widehat{Y}$ is any solution to Φ^X then $\Delta^{X \oplus \widehat{Y}}$ is a solution to X.
(4) P is *strongly Weihrauch reducible* to Q, written $\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q}$, if there are Turing functionals Φ and Δ such that if X is any instance of P then Φ^X is an instance of Q, and if $\widehat{Y}$ is any solution to Φ^X then $\Delta^{\widehat{Y}}$ is a solution to X.

All of these reductions express the idea of taking a problem, P, and computably (even uniformly computably, in the case of $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$) transforming it into another problem, Q, in such a way that being able to solve the latter computably (uniformly computably) tells us how to solve the former. This is a natural idea, and indeed, more often than not an implication $\mathsf{Q} \to \mathsf{P}$ over RCA_0 (or at least, over ω-models of RCA_0) is a formalization of some such reduction. The strong versions above may appear more contrived, since it does not seem reasonable to deliberately bar access to the instance of the problem one is working with. Yet commonly, in a reduction of the above sort, the "backward" computation from $\widehat{Y}$ to Y turns out not to reference the original instance. Frequently, it is just the identity.

Let $\mathsf{P} \leq_{\omega} \mathsf{Q}$ denote that every ω-model of $\mathsf{RCA}_0 + \mathsf{Q}$ is a model of P. It is easy to see that the following implications hold:

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