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The countable admissible ordinal equivalence relation ☆

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A R T I C L E I N F O

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A B S T R A C T

Let F_{ω_1} be the countable admissible ordinal equivalence relation defined on ${}^\omega 2$ by $x\ F_{\omega_1}\ y$ if and only if $\omega_1^x = \omega_1^y$. Some invariant descriptive set theoretic properties of F_{ω_1} will be explored using infinitary logic in countable admissible fragments as the main tool. Marker showed F_{ω_1} is not the orbit equivalence relation of a continuous action of a Polish group on ${}^\omega 2$. Becker stengthened this to show F_{ω_1} is not even the orbit equivalence relation of a $\mathbf{\Delta}_1^1$ action of a Polish group. However, Montalbán has shown that F_{ω_1} is $\mathbf{\Delta}_1^1$ reducible to an orbit equivalence relation of a Polish group action, in fact, F_{ω_1} is classifiable by countable structures. It will be shown here that F_{ω_1} must be classified by structures of high Scott rank. Let E_{ω_1} denote the equivalence of order types of reals coding well-orderings. If E and F are two equivalence relations on Polish spaces X and Y, respectively, $E \leq_{\mathrm{a}\mathbf{\Delta}_1^1} F$ denotes the existence of a $\mathbf{\Delta}_1^1$ function $f : X \to Y$ which is a reduction of E to F, except possibly on countably many classes of E. Using a result of Zapletal, the existence of a measurable cardinal implies $E_{\omega_1} \leq_{\mathrm{a}\mathbf{\Delta}_1^1} F_{\omega_1}$. However, it will be shown that in Gödel's constructible universe L (and set generic extensions of L), $E_{\omega_1} \leq_{\mathrm{a}\mathbf{\Delta}_1^1} F_{\omega_1}$ is false. Lastly, the techniques of the previous result will be used to show that in L (and set generic extensions of L), the isomorphism relation induced by a counterexample to Vaught's conjecture cannot be $\mathbf{\Delta}_1^1$ reducible to F_{ω_1}. This shows the consistency of a negative answer to a question of Sy-David Friedman.

1. Introduction

If $x \in {}^\omega 2$, ω_1^x denotes the supremum of the order types of x-recursive well-orderings on ω. Moreover, ω_1^x is also the minimum ordinal height of admissible sets containing x as an element. The latter definition will be more relevant for this paper.

The eponymous countable admissible ordinal equivalence relation, denoted by F_{ω_1}, is defined on ${}^\omega 2$ by:

$$x\ F_{\omega_1}\ y \Leftrightarrow \omega_1^x = \omega_1^y$$

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It is a Σ_1^1 equivalence relation with all classes $\mathbf{\Delta}_1^1$. Moreover, F_{ω_1} is a thin equivalence relation, i.e., it has no perfect set of inequivalence elements. Some further properties of F_{ω_1} as an equivalence relation will be established in this paper.

Some basic results in admissibility theory and infinitary logic that will be useful throughout the paper will be reviewed in Section 2. This section will cover briefly topics such as KP, admissible sets, Scott ranks, and the Scott analysis. In this section, aspects of Barwise's theory of infinitary logic in countable admissible fragments, which will be the main tool in many arguments, will be reviewed. As a example of an application, a proof of a theorem of Sacks (Theorem 2.16), which establishes that every countable admissible ordinal is of the form ω_1^x for some $x \in {}^\omega 2$, will be given. This proof serves as a template for other arguments. Sacks' theorem also explains why it is appropriate to call F_{ω_1} the "countable admissible ordinal equivalence relation".

There have been some early work on whether F_{ω_1} satisfies certain properties of equivalence relations related to generalization of Vaught's conjecture. For example, Marker in [15] has shown that F_{ω_1} is not induced by a continuous action of a Polish group on the Polish space ${}^\omega 2$. Becker in [3], page 782, strengthened this to show that the equivalence relation F_{ω_1} is not an orbit equivalence relation of a $\mathbf{\Delta}_1^1$ group action of a Polish group. A natural question following these results would be whether F_{ω_1} is $\mathbf{\Delta}_1^1$ reducible to equivalence relations induced by continuous or $\mathbf{\Delta}_1^1$ actions of Polish groups. If such reductions do exist, another question could be what properties must these reductions have.

In Section 3, F_{ω_1} will be shown to be $\mathbf{\Delta}_1^1$ reducible to a continuous action of S_∞, i.e., it is classifiable by countable structures. An explicit Δ_1^1 classification of F_{ω_1} by countable structures in the language with a single binary relation symbol, due to Montalbán, will be provided. The classification of F_{ω_1} will use an effective construction of the Harrison linear ordering. This classification, denoted f, has the additional property that for all $x \in {}^\omega 2$, $\mathrm{SR}(f(x)) = \omega_1^x + 1$. This example was provided by Montalbán through communication with Marks and the author.

The explicit classification, f, mentioned above has images that are structures of high Scott rank. In Section 4, it will be shown that this is a necessary feature of all classification of F_{ω_1} by countable structures. The lightface version of the main result of this section is the following:

Theorem 4.2. *Let $\mathscr{L}$ be a recursive language. Let $S(\mathscr{L})$ denote the set of reals that code $\mathscr{L}$-structures on ω. If $f : {}^\omega 2 \to S(\mathscr{L})$ is a Δ_1^1 function such that $x \; F_{\omega_1} \; y$ if and only if $f(x) \cong_{\mathscr{L}} f(y)$, then for all x, $SR(f(x)) \geq \omega_1^x$.*

The more general form considers reductions that are $\Delta_1^1(z)$ and involves a condition on the admissible spectrum of z. Intuitively, Theorem 4.2 (in its lightface form as stated above) asserts that any potential classification of F_{ω_1} must have high Scott rank in the sense that the image of any real under the reduction is a structure of high Scott rank. High Scott rank means that $\mathrm{SR}(f(x))$ is either ω_1^x or $\omega_1^x + 1$.

Section 5 is concerned with a weak form of reduction of equivalence relations, invented by Zapletal, called the almost $\mathbf{\Delta}_1^1$ reduction. If E and F are two $\mathbf{\Sigma}_1^1$ equivalence relations on Polish space X and Y, respectively, then E is almost $\mathbf{\Delta}_1^1$ reducible to F (in symbols: $E \leq_{\mathrm{a}\mathbf{\Delta}_1^1} F$) if and only there is a $\mathbf{\Delta}_1^1$ function $f : X \to Y$ and a countable set A such that if x and y are not E-related to any elements of A, then $x \; E \; y$ if and only if $f(x) \; F \; f(y)$.

An almost Borel reduction is simply a reduction that may fail on countably many classes. Often $\mathbf{\Sigma}_1^1$ equivalence relations may have a few unwieldy classes. The almost Borel reduction is especially useful since it can be used to ignore these classes. One example of such a $\mathbf{\Sigma}_1^1$ equivalence relation is E_{ω_1} which is the isomorphism relation of well-orderings with a single class of non-well-orderings. It is defined on ${}^\omega 2$ by:

$$x \; E_{\omega_1} \; y \Leftrightarrow (x, y \notin WO) \vee (\mathrm{ot}(x) = \mathrm{ot}(y))$$

E_{ω_1} is a thin Σ_1^1 equivalence with one Σ_1^1 class and all the other classes are $\mathbf{\Delta}_1^1$.

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