Annals of Pure and Applied Logic 168 (2017) 1308–1334

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

On expansions of the real field by complex subgroups

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A R T I C L E   I N F O

Article history:
Received 5 February 2016
Received in revised form 21 December 2016
Accepted 2 January 2017
Available online 6 January 2017

MSC:
03C64
03C10
14P10
20E99

Keywords:
Model theory
Real field
Multiplicative subgroups
Finite rank
Complex numbers

A B S T R A C T

We construct a class of finite rank multiplicative subgroups of the complex numbers such that the expansion of the real field by such a group is model-theoretically well-behaved. As an application we show that a classification of expansions of the real field by cyclic multiplicative subgroups of the complex numbers due to Hieronymi does not even extend to expansions by subgroups with two generators.

1. Introduction

Let $\overline{\mathbb{R}} := (\mathbb{R}, <, +, \cdot)$ be the real field. This paper contributes to the classification of expansions of $\overline{\mathbb{R}}$ by finite rank multiplicative subgroups S of complex numbers. Here we identify $\mathbb{C}$ with $\mathbb{R}^2$ as usual and consider expansions of $\overline{\mathbb{R}}$ by a binary predicate for the multiplicative subgroup. This is not the first time such structures have been studied. Belegradek and Zilber [1] and independently Günaydın [8] initiated the study of such expansions by fully determining the model theory of such expansions when S is a finite rank subgroup of the unit circle $\mathbb{S}^1$. Using this work Hieronymi [10] established that if S is a cyclic subgroup of $\mathbb{C}$ (not necessarily a subgroup of $\mathbb{S}^1$), then exactly one of the following statements holds:

(i) $(\overline{\mathbb{R}}, S)$ defines $\mathbb{Z}$,
(ii) $(\overline{\mathbb{R}}, S)$ is d-minimal, or

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http://dx.doi.org/10.1016/j.apal.2017.01.002
0168-0072/

(iii) every open definable set in $(\overline{\mathbb{R}}, S)$ is semialgebraic.

An ordered structure $\mathcal{R}$ is *d-minimal* if for every $M \equiv \mathcal{R}$, every subset of M is a disjoint union of open intervals and finitely many discrete sets. More is known: By Theorem 1.3 in Günaydın and Hieronymi [9] every finite rank subgroup of $\mathbb{S}^1$ satisfies (iii), and therefore this classification can be extended to include such groups. This leads naturally to the question whether this holds true for arbitrary finite rank subgroups. The main result of this paper is a negative answer to this question.

Theorem A. *Let Γ be a finite rank subgroup of $\mathbb{S}^1$ which is dense in $\mathbb{S}^1$, let $\Delta = \varepsilon^{\mathbb{Z}}$ for $\varepsilon \in \mathbb{R}^{>0}$, and set $S = \Gamma\Delta$. Then every subset of $\mathbb{R}^m$ definable in $(\overline{\mathbb{R}}, S)$ is a Boolean combination of sets of the form*

$$\{x \in \mathbb{R}^m : \exists y \in S^n \text{ s.t. } (x, y) \in W\}$$

for some semialgebraic set $W \subseteq \mathbb{R}^{m+2n}$. Moreover, every open definable set in $(\overline{\mathbb{R}}, S)$ is definable in $(\overline{\mathbb{R}}, \Delta)$.

It is not hard to see that $(\overline{\mathbb{R}}, S)$ does not satisfy any of (i)–(iii). First note that $(\overline{\mathbb{R}}, S)$ defines both Γ and Δ. If $(\overline{\mathbb{R}}, S)$ defines $\mathbb{Z}$, then by [11, (37.6)], $(\overline{\mathbb{R}}, S)$ defines every projective subset of $\mathbb{R}$. However, it can be checked that $(\overline{\mathbb{R}}, S)$ does not define every projective subset of $\mathbb{R}$. For example, if S is countable, then every subset of $\mathbb{R}$ which is definable in $(\overline{\mathbb{R}}, S)$ is a Boolean combination of F_σ sets by Theorem A. The projection of S onto the real line is a definable set that is dense and codense, so $(\overline{\mathbb{R}}, S)$ is not d-minimal. Lastly, the complement of Δ in the real line is open and definable in $(\overline{\mathbb{R}}, S)$, but is not semialgebraic. By picking Γ to be the group generated by $e^{i\pi\varphi}$ for some irrational $\varphi \in \mathbb{R}^{>0}$, we see that the above classification fails even for multiplicative subgroups generated by two elements.

The fact that the sets definable in $(\overline{\mathbb{R}}, S)$ have the form given in Theorem A is proved in Section 5.4. We call this property quantifier reduction. The fact that every open definable set in $(\overline{\mathbb{R}}, S)$ is definable in $(\overline{\mathbb{R}}, \Delta)$ is proved in Section 6. In addition to Theorem A we will also give an axiomatization for such structures in Section 4. Let Γ be a dense subgroup of $\mathbb{S}^1$, let $\Delta = \varepsilon^{\mathbb{Z}}$ for some $\varepsilon \in \mathbb{R}^{>0}$, and set $S = \Gamma\Delta$. Since both Γ and Δ are definable in $(\overline{\mathbb{R}}, S)$, we will consider the structure $(\overline{\mathbb{R}}, \Gamma, \Delta)$ instead. We further expand this structure by constant symbols for each element in $\mathrm{Re}(\Gamma) \cup \mathrm{Im}(\Gamma)$ and Δ.

Theorem B. *Let K be a real closed field. Let G be a dense subgroup of $\mathbb{S}^1(K)$ and let $\gamma \mapsto \gamma' : \Gamma \to G$ be a group homomorphism. For $\gamma \in \Gamma$ with $\gamma = (\alpha, \beta)$, let α' and β' be such that $\gamma' = (\alpha', \beta')$. Let A be a subgroup of $K^{>0}$ with a group homomorphism $\delta \mapsto \delta' : \Delta \to A$ such that*

(i) ε' is the smallest element of A greater than 1, and
(ii) for every $k \in K^{>0}$, there is $a \in A$ such that $a \leq k < a\varepsilon'$.

Then

$$(K, G, A, (\delta')_{\delta\in\Delta}, (\gamma')_{\gamma\in\Gamma}) \equiv (\overline{\mathbb{R}}, \Gamma, \Delta, (\delta)_{\delta\in\Delta}, (\gamma)_{\gamma\in\Gamma})$$

if and only if:

1. *for every $\gamma \in \Gamma$ and $n \in \mathbb{Z}^{>0}$, γ is an nth power in Γ if and only if γ' is an nth power in G;*
2. *for all primes p, $[p]\Gamma = [p]G$;*
3. *for all $n \in \mathbb{Z}^{>0}$, all polynomials $Q(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$, and all tuples $(\gamma_1\delta_1, \dots, \gamma_n\delta_n)$ of elements of $\Gamma\Delta$,*

$$Q(\mathrm{Re}(\gamma_1\delta_1), \dots, \mathrm{Re}(\gamma_n\delta_n)) > 0 \text{ if and only if } Q(\mathrm{Re}(\gamma_1'\delta_1'), \dots, \mathrm{Re}(\gamma_n'\delta_n')) > 0;$$