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Interaction graphs: Graphings

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A B S T R A C T

In two previous papers [33,37], we exposed a combinatorial approach to the program of Geometry of Interaction, a program initiated by Jean-Yves Girard [16]. The strength of our approach lies in the fact that we interpret proofs by simpler structures – graphs – than Girard's constructions, while generalising the latter since they can be recovered as special cases of our setting. This third paper extends this approach by considering a generalisation of graphs named *graphings*, which is in some way a *geometric realisation* of a graph on a measured space. This very general framework leads to a number of new models of multiplicative-additive linear logic which generalise Girard's geometry of interaction models and opens several new lines of research. As an example, we exhibit a family of such models which account for second-order quantification without suffering the same limitations as Girard's models.

1. Introduction

1.1. Context

Geometry of interaction. This research program was introduced by Girard [13,16] after his discovery of linear logic [12]. In a first approximation, it aims at defining a semantics of proofs that accounts for the dynamics of cut-elimination. Namely, the geometry of interaction models differ from usual (denotational) semantics in that the interpretation of a proof π and its normal form ρ are not equal, but one has a way of computing the interpretation of the normal form ρ from the interpretation of the proof π (illustrated in Fig. 1). As a consequence, a geometry of interaction models not only proofs – programs – but also their normalization – their execution. This semantical counterpart to the cut-elimination procedure was called the *execution formula* by Girard in his first papers about geometry of interaction [15,14,17], and it is a way of computing

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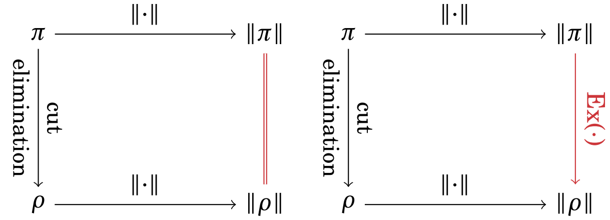

(a) Denotational Semantics (b) Geometry of Interaction

Fig. 1. Denotational semantics vs geometry of interaction.

the solution to the so-called *feedback equation*. This equation turned out to have a more general solution [18], which lead Girard to the definition of a geometry of interaction in the hyperfinite factor [19].

Geometry of Interaction, however, is not only about the interpretation of proofs and their dynamics, but also about reconstructing logic around this semantical counterpart to the cut-elimination procedure. This means that logic arises from the dynamics and interactions of proofs – programs –, as a syntactical description of the possible behaviours of proofs – programs. This aspect of the geometry of interaction program has been less studied than the proof interpretation part.

We must also point out that geometry of interaction has been successful in providing tools for the study of computational complexity. The fact that it models the execution of programs explains that it is well suited for the study of complexity classes in time [5,27], as well as in space [4,3]. It was also used to explain [23] Lamping's optimal reduction of lambda-calculus [28].

Interaction graphs. They were first introduced [33] to define a combinatorial approach to Girard's geometry of interaction in the hyperfinite factor [19]. The main idea was that the execution formula – the counterpart of the cut-elimination procedure – can be computed as the set of alternating paths between graphs, and that the measurement of interaction defined by Girard using the Fuglede–Kadison determinant [11] can be computed as a measurement of a set of cycles.

The setting was then extended to deal with additive connectives [37], showing by the way that the constructions were a combinatorial approach not only to Girard's hyperfinite GoI construction but also to all the earlier constructions [13,15,14,17]. This result could be obtained by unveiling a single geometrical property, which we called the *trefoil property*, upon which all the constructions of geometry of interaction introduced by Girard are founded. This property, which can be understood as a sort of associativity, suggests that computation – as modelled by geometry of interaction – is closely related to algebraic topology.

This paper takes another direction though: based on ideas that appeared in the author's PhD thesis [34], it extends the setting of graphs by considering a generalisation of graphs named *graphings*, which is in some way a *geometric realisation* of a graph. This very general framework leads to a number of new models of multiplicative-additive linear logic which generalise Girard's geometry of interaction models and opens several new lines of research. As an example, we exhibit a family of such models which account for second-order quantification without suffering the same limitations as Girard's models.

1.2. Outline

We introduce in this paper a family of models which generalises and axiomatizes the notion a GoI model. This systematic approach is obtained by extending previous work [33,37]. While these previous constructions built models in which the objects under study were directed weighted graphs, we here consider a measure-theoretic generalisation of such graphs named *graphings*. The resulting construction yields a very rich hierarchy of models parametrized by two monoids: the *weight monoid* and the so-called *microcosm*.

A weight monoid is nothing more than a monoid which elements will be used to give weights to edges of the graphs. A microcosm, on the other hand, is a monoid of measurable maps, i.e. it is a subset of $\mathscr{M}(X)$, the

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