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A micrological study of negation

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ABSTRACT

Tensorial logic is a primitive logic of tensor and negation which refines linear logic by relaxing the hypothesis that linear negation is involutive. Thanks to this mild modification, tensorial logic provides a type-theoretic account of game semantics where innocent strategies are portrayed as temporal refinements of traditional proof-nets in linear logic. In this paper, we study the algebraic and combinatorial structure of negation in a non-commutative variant of tensorial logic. The analysis is based on a 2-categorical account of dialogue categories, which unifies tensorial logic with linear logic, and discloses a primitive symmetry between proofs and anti-proofs. The micrological analysis of tensorial negation reveals that it can be decomposed into a series of more elementary components: an adjunction $L \dashv R$ between the left and right negation functors L and R; a pair of linear distributivity laws $\kappa^{\oslash}$ and $\kappa^{\odot}$ which refines the linear distributivity law between $\otimes$ and $⅋$ in linear logic, and generates the Opponent and Proponent views of innocent strategies between dialogue games; a pair of axiom and cut combinators adapted from linear logic; an involutive change of frame $(-)^*$ reversing the point of view of Prover and of Denier on the logical dispute, and reversing the polarity of moves in the dialogue game associated to the tensorial formula.

1. Introduction

The general ambition of our work in tensorial logic is to lift the ideas and constructions of linear logic to this more primitive logic where tensorial negation $A \mapsto \neg A$ is not required to be involutive. This research programme bumps against the apparent objection that tensorial logic is an "intuitionistic" logic and thus lacks the familiar "classical" symmetries of linear logic [6,8]. We show how to circumvent this objection by establishing that the "classical" symmetries of linear logic are derived from a more primitive symmetry of logic which governs both tensorial logic and linear logic. The action of this symmetry

$$Prover \leftrightarrow Denier$$

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is to interchange the two sides *Prover* and *Denier* of a logical dispute. We advocate that, seen from a 2-categorical point of view, this symmetry internal to logic coincides with the 2-dimensional duality internal to category theory

$$\mathscr{C} \leftrightarrow \mathscr{C}^{op}$$

between a category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$. As such, the symmetry is "proto-logical" and works in the same way for classical logic and for intuitionistic logic.

In this introduction, we explain how the symmetry *Prover* $\leftrightarrow$ *Denier* emerged from our attempts to extend the "classical" symmetries of linear logic to an intuitionistic logic like tensorial logic. Since our approach to proof theory is mainly algebraic, we find convenient to start from the notion of $*$-autonomous category which provides the categorical counterpart of linear logic. Moreover, in order to work on the most precise and general notion of negation, we focus in this paper on *non-commutative* variants of linear logic and tensorial logic. We leave the study of pivotal, ribbon or symmetric versions of tensorial logic to subsequent papers.

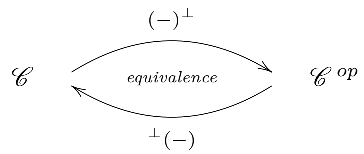
Recall that a non-symmetric $*$-autonomous category is traditionally defined as a biclosed monoidal category $\mathscr{C}$ equipped with a dualizing object $\bot$, see [2,10] for details. An object $\bot$ of a biclosed monoidal category $\mathscr{C}$ is called *dualizing* when the two canonical morphisms

$$x \longrightarrow \bot \multimapinv (x \multimap \bot) \qquad\qquad x \longrightarrow (\bot \multimapinv x) \multimap \bot$$

transporting x into its double negation are isomorphisms, for every object x of the dialogue category. The terminology of "dualizing object" comes from the fact that the two negation functors

$$\begin{array}{ccccccccc} (-)^{\bot} & : & x & \mapsto & x \multimap \bot & : & \mathscr{C} & \longrightarrow & \mathscr{C}^{op} \\ {}^{\bot}(-) & : & x & \mapsto & \bot \multimapinv x & : & \mathscr{C}^{op} & \longrightarrow & \mathscr{C} \end{array}$$

define in that case an equivalence



(1)

between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$. This establishes that every $*$-autonomous category $\mathscr{C}$ is self-dual, in the technical sense that it is equivalent to its opposite category.

This specific definition of $*$-autonomous category is prevailing today. However, one is entitled to complain about the fact that this accepted definition is not sufficiently "symmetric" for the following reason: it starts from the conjunction $\otimes$ and from the two implications $\multimap$ and $\multimapinv$ provided by the biclosed monoidal category, rather than from the conjunction $\otimes$ and from the disjunction $\parr$ provided by the traditional presentation of linear logic. From a purely aesthetic point of view, this non-symmetric presentation of a perfectly self-dual logic like linear logic looks awkward. One thus wonders whether the formulation may be replaced by a properly symmetric one. A preliminary step in that direction is to observe is that there are two canonical ways to define the disjunction in a biclosed monoidal category:

$$\begin{array}{lclcl} x \parr_1 y & := & \bot \multimapinv ((y \multimap \bot) \otimes (x \multimap \bot)) & := & {}^{\bot}(y^{\bot} \otimes x^{\bot}) \\ x \parr_2 y & := & ((\bot \multimapinv y) \otimes (\bot \multimapinv x)) \multimap \bot & := & ({}^{\bot}y \otimes {}^{\bot}x)^{\bot} \end{array}$$

and that the two disjunctions $\parr_1$ and $\parr_2$ are isomorphic when the object $\bot$ is dualizing. The reason is that the two canonical morphisms