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On non-self-referential fragments of modal logics

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ABSTRACT

Justification logics serve as "explicit" modal logics in a way that, formula ϕ is a modal theorem if and only if there is a justification theorem, called a *realization* of ϕ, gained by replacing modality occurrences in ϕ by (justification) terms with structures explicitly explaining their evidential contents. In justification logics, terms stand for justifications of (propositions expressed by) formulas, and as a kind of atomic terms, constants stand for that of (justification) axioms. Kuznets has shown that in order to realize (i.e., offer a realization of) some modal theorems, it is necessary to employ a self-referential constant, that is, a constant that stands for a justification of an axiom containing an occurrence of the constant itself. Based on existing works, including some of the author's, this paper treats the collection of modal theorems that are non-self-referentially realizable as a fragment (called *non-self-referential fragment*) of the modal logic, and verifies: (1) that fragment is not closed in general under *modus ponens*; and (2) that fragment is not "conservative" in general when going from a smaller modal logic to a larger one.

1. Introduction

1.1. Modal logic

Modal language is propositional language equipped with modality (also called modal operator) $\Box$. The dual of $\Box$, usually written as $\Diamond$, can be seen as defined in classical setting. Depending on the logic that uses the modal language, modality $\Box$ has variant meanings, like "it is necessary that" (in modal logic), "it is known/believed that" (in epistemic/doxastic logic), "it is provable that" (in provability logic), etc. The name "modal logic" is also widely used to refer to the whole class of logics using such a language.

In this paper, the word *modal logic* refers to the class of logics using the language defined by

$$\phi ::= \bot | p | \phi \to \phi | \Box\phi,$$

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[2] Part of this work was done when the author was in a Ph.D. program in Computer Science at the Graduate Center, City University of New York, and was included in the author's thesis [33].

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Table 1
Axiom schemes of five modal logics.

Name	Axiom schemes	Employed by Logic
Prop	classical propositional axiom schemes	All
K	$\Box(\alpha\to\beta)\to(\Box\alpha\to\Box\beta)$	All
D	$\Box\bot\to\bot$	D
T	$\Box\phi\to\phi$	T, S4
4	$\Box\phi\to\Box\Box\phi$	K4, S4

where p is a propositional atom.

For the propositional base, we assume countable infinitely many propositional atoms, and by a *prime formula*, we mean either a propositional atom or a $\bot$. Chosen as primitive propositional connectives are $\bot$ (falsehood) and $\to$ (implication), which works well for classical settings as other connectives can be routinely defined as abbreviations (e.g., $\neg\phi$ is an abbreviation of $\phi\to\bot$).

For modalities, we take $\Box$ as primitive, and use $\Diamond$ only as an abbreviation of $\neg\Box\neg$. In this paper, we will focus on five modal logics, K, D, T, K4, and S4. They share the language defined above, classical propositional axiom schemes, rule schemes (MP) (*modus ponens*) and (Nec) (necessitation, if ϕ is a theorem then so is $\Box\phi$), but differ with respect to axiom schemes concerning modalities. Axiom schemes of these five logics are summarized in Table 1.

It is well-known that K is the smallest logic among all the five, D is smaller than S4, the pair T and K4 are incomparable, etc.. Actually, we can display relations between them as follows, where logics to the down/left are smaller than that to the up/right.

$$\begin{tikzcd}
\mathsf{K4} \arrow[rr, dash] \arrow[d, dash] & & \mathsf{S4} \arrow[d, dash] \\
\mathsf{K} \arrow[r, dash] & \mathsf{D} \arrow[r, dash] & \mathsf{T}
\end{tikzcd}$$

For more about modal logics, please refer to textbooks [10,5].

Some notational conventions here. $\mathbb{N}$ is the set of natural numbers, and $\mathbb{N}^+$ is $\mathbb{N}\backslash\{0\}$. By $\alpha\equiv\beta$ we mean symbol strings α and β are literally identical. For any prefixing unary operator ∇, define $\nabla^0\phi :\equiv \phi$ and $\nabla^{n+1}\phi :\equiv \nabla\nabla^n\phi$ for any $n\in\mathbb{N}$ (for instance, $\Box^3 p\equiv\Box\Box\Box p$). A (fragment of a) logic is understood as the set of all its theorems, and hence $\phi\in\mathsf{X}$ means "ϕ is an X-theorem." By $\Gamma\vdash_{\mathsf{X}}\phi$ we mean that there is an X-derivation of ϕ with hypotheses from Γ. If $\varnothing\vdash_{\mathsf{X}}\phi$ (i.e., ϕ is an X-theorem), we also write $\vdash_{\mathsf{X}}\phi$. Denote the set $\{\mathsf{K},\mathsf{D},\mathsf{T},\mathsf{K4},\mathsf{S4}\}$ by 5ML, and hence $\mathsf{X}\in\mathsf{5ML}$ means "X is one of the five modal logics K, D, T, K4, and S4."

1.2. Justification logic

Justification logic can be seen as "explicit version" of modal logic. Its language is also propositionally based, but instead of merely $\Box$, it has constructive modalities. For instance, instead of $\Box\phi$ with meaning like "ϕ is believed" (or "ϕ is provable"), we have $x_1+(c\cdot x_2):\psi$, with meaning like "ψ is believed for the justification $x_1+(c\cdot x_2)$" (or "ψ is proved by the proof $x_1+(c\cdot x_2)$"), where c, x_1, x_2 stand for atomic justifications and operators $+$ and $\cdot$ stand for specific operations on justifications.

This paper focuses on five justification logics, J, JD, JT, J4, and LP. They each serves as the explicit version of a logic in 5ML, as indicated by their names.[3] Being historically the first among them, LP appeared in Artemov's [1,2]. The other four are from Brezhnev's [6].

[3] As an exception, the explicit version of S4 is called *the Logic of Proofs* (LP) for its special position in formalizing provability semantics of S4, and then of intuitionistic propositional logic.