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Covering the recursive sets ☆

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A B S T R A C T

We give solutions to two of the questions in a paper by Brendle, Brooke-Taylor, Ng and Nies. Our examples derive from a 2014 construction by Khan and Miller as well as new direct constructions using martingales.

At the same time, we introduce the concept of i.o. subuniformity and relate this concept to recursive measure theory. We prove that there are classes closed downwards under Turing reducibility that have recursive measure zero and that are not i.o. subuniform. This shows that there are examples of classes that cannot be covered with methods other than probabilistic ones. It is easily seen that every set of hyperimmune degree can cover the recursive sets. We prove that there are both examples of hyperimmune-free degree that can and that cannot compute such a cover.

1. Introduction

An important theme in set theory has been the study of cardinal characteristics. As it turns out, in the study of these there are certain analogies with recursion theory, where the recursive sets correspond to sets in the ground model. Recently, Brendle, Brooke-Taylor, Ng and Nies [1] point out analogies between

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cardinal characteristics and the study of algorithmic randomness. We address two questions raised in this paper that are connected to computing covers for the recursive sets.

In the following, we will assume that the reader is familiar with various notions from computable measure theory, in particular, with the notions of Martin-Löf null, Schnorr null and Kurtz null set. For background on these notions we refer the reader to the books of Calude [2], Downey and Hirschfeldt [5], Li and Vitányi [14] and Nies [16].

Our notation from recursion theory is mostly standard, except for the following: The natural numbers are denoted by ω, 2^ω denotes the Cantor space and $2^{<\omega}$ the set of all finite binary sequences. $\mathbb{R}^{\geqslant 0}$ denotes the set of those real numbers which are not negative. We denote the concatenation of strings σ and τ by $\sigma\tau$. The notation $\sigma \sqsubseteq \tau$ denotes that the finite string σ is an initial segment of the (finite or infinite) string τ. We identify sets $A \subseteq \omega$ with their characteristic sequences, and $A\restriction n$ denotes the initial segment $A(0)\dots A(n-1)$. We use λ to denote the empty string. Throughout, μ denotes the Lebesgue measure on 2^ω. We write $a \simeq b$ if either both sides are undefined, or they are both defined and equal. We let $\mathrm{Parity}(x) = 0$ if x is even, and $\mathrm{Parity}(x) = 1$ if x is odd.

Definition 1. A function $M : 2^{<\omega} \to \mathbb{R}^{\geqslant 0}$ is a *martingale* if for every $x \in 2^{<\omega}$, M satisfies the averaging condition

$$2M(\sigma) = M(\sigma 0) + M(\sigma 1).$$

A martingale M *succeeds on* a set A if

$$\limsup_{n\to\infty} M(A\restriction n) = \infty.$$

The class of all sets on which M succeeds is denoted by $S[M]$.

For more background material on recursive martingales we refer the reader to the above mentioned textbooks [2,5,14,16]. The following definition is taken from Rupprecht [19,20].

Definition 2. An oracle A is *Schnorr covering* if the union of all Schnorr null sets is Schnorr null relative to A. An oracle A is *weakly Schnorr covering* if the set of recursive reals is Schnorr null relative to A. For the latter, we will also say that A Schnorr covers REC.

Definition 3. A *Kurtz test* relative to A is an A-recursive sequence of closed-open sets G_i such that each G_i has measure at most 2^{-i}; these closed-open sets are given by explicit finite lists of strings and they consist of all members of $\{0, 1\}^\omega$ extending one of the strings. Note that $i \to \mu(G_i)$ can be computed relative to A. The intersection of a Kurtz test (relative to A) is called a Kurtz null set (relative to A). An oracle A is *Kurtz covering* if there is an A-recursive array $G_{i,j}$ of closed-open sets such that each i-th component is a Kurtz test relative to A and every unrelativised Kurtz test describes a null-set contained in $\cap_j G_{i,j}$ for some i; A is *weakly Kurtz covering* if there is such an array and each recursive sequence is contained in some A-recursive Kurtz null set $\cap_j G_{i,j}$.

Brendle, Brooke-Taylor, Ng and Nies [1] called the notion of (weakly) Schnorr covering in their paper (weakly) Schnorr *engulfing*. In this paper, we will use the original terminology of Rupprecht [19,20]. We have analogous notions for the other notions of effective null sets. As mentioned above, a set A is weakly Kurtz covering if the set of recursive reals is Kurtz null relative to A. We also have Baire category analogues of these notions of covering: A set A is *weakly meager covering* if it computes a meager set that contains all recursive reals; more precisely, A is weakly meager covering iff there is an A-recursive function f mapping each binary string σ to an extension $f(\sigma)$ such that every recursive sequence B has only finitely many prefixes