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Randomness for computable measures and initial segment complexity

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ABSTRACT

We study the possible growth rates of the Kolmogorov complexity of initial segments of sequences that are random with respect to some computable measure on 2^ω, the so-called proper sequences. Our main results are as follows: (1) We show that the initial segment complexity of a proper sequence X is bounded from below by a computable function (that is, X is complex) if and only if X is random with respect to some computable, continuous measure. (2) We prove that a uniform version of the previous result fails to hold: there is a family of complex sequences that are random with respect to a single computable measure such that for every computable, continuous measure μ, some sequence in this family fails to be random with respect to μ. (3) We show that there are proper sequences with extremely slow-growing initial segment complexity, that is, there is a proper sequence the initial segment complexity of which is infinitely often below every computable function, and even a proper sequence the initial segment complexity of which is dominated by all computable functions. (4) We prove various facts about the Turing degrees of such sequences and show that they are useful in the study of certain classes of pathological measures on 2^ω, namely diminutive measures and trivial measures.

1. Introduction

The Levin–Schnorr Theorem establishes the equivalence of a certain measure-theoretic notion of typicality for infinite sequences (known as Martin-Löf randomness) with a notion of incompressibility given in terms of Kolmogorov complexity. Although the Levin–Schnorr Theorem is usually formulated for sequences that are random with respect to the Lebesgue measure on 2^ω, it is well known that the theorem can be generalized to hold for any computable probability measure on 2^ω. More specifically, a sequence $X \in 2^\omega$ is Martin-Löf random with respect to a computable measure μ if and only if the initial segment complexity of $X{\upharpoonright}n$ is

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bounded from below by $-\log \mu(X\restriction n)$. Thus we see that certain values of the measure μ constrain the possible values of the initial segment complexities of the μ-random sequences.

In this study, we further explore the interaction between computable measures and the initial segment complexity of the sequences that are random with respect to these measures (hereafter, we will refer to those sequences that are random with respect to a computable measure as *proper* sequences, following the terminology of Zvonkin and Levin [30]). In the first half of this article we focus on the relationship between a class of sequences known as *complex sequences* and those sequences that are random with respect to a computable, continuous measure. First studied systematically by Kjos-Hanssen et al. [15] (but also studied earlier by Kanovič [13]), complex sequences are those sequences whose initial segment complexities are bounded below by some computable function. We characterize the complex proper sequences as the sequences that are random with respect to some computable *continuous* measure. This is done by studying the "removability" of μ-atoms, that is, sequences X such that $\mu(\{X\}) > 0$: We show that if a sequence X is complex and random with respect to some computable measure μ, we can define a computable, continuous measure ν such that X is random with respect to ν by removing the atoms from μ while preserving X's randomness. It is natural to ask whether this removal of atoms can always be carried out while preserving *all* non-atomic random sequences simultaneously, again assuming that all of these random sequences are complex. We show that this is not the case.

Using this characterization of complex sequences through computable continuous measures, we establish new results on the relationship between the notions of avoidability, hyperavoidability, semigenericity, and not being random for any computable, continuous measure. More specifically, when restricted to the collection of proper sequences, we show that these four notions are equivalent to being complex. We also study the granularity of a computable, continuous measure μ and show that the inverse of the granularity function provides a uniform lower bound for the initial segment complexity of μ-random sequences.

In the second half of this article we turn our attention to atomic computable measures, that is, computable measures μ that have μ-atoms. First, we study atomic measures μ with the property that every μ-random sequence is either a μ-atom or is complex. We show that for such measures μ, even though the initial segment complexity of each non-atom μ-random sequence is bounded from below by some computable function, there is in general no uniform computable lower bound for every non-atom μ-random sequence. Next, we construct a computable atomic measure μ with the property that the initial segment complexity of each μ-random sequence dominates no computable function, and a computable atomic measure ν with the property that the initial segment complexity of each ν-random sequence is dominated by all computable functions. The former sequences are called *infinitely often anti-complex*, while the latter are known simply as *anti-complex*.

Lastly, we study two specific kinds of atomic measures: diminutive measures and trivial measures. Here, a measure μ is *trivial* if $\mu(\mathrm{Atoms}_\mu) = 1$, and diminutive measures are defined as follows. First, for $\mathcal{C} \subseteq 2^\omega$, $\mathcal{C}$ is *diminutive* if it does not contain a computably perfect subclass. Let μ be a computable measure, and let $(\mathcal{U}_i)_{i\in\omega}$ be a universal μ-Martin-Löf test. Then we say that μ is *diminutive* if $\mathcal{U}_i^c$ is a diminutive Π^0_1 class for every i. We show that while every computable trivial measure is diminutive, the converse does not hold. The proof of this last statement gives an alternative, priority-free proof of a result by Kautz [14] showing that there is a computable, non-trivial measure μ such that there is no Δ^0_2, non-computable $X \in \mathrm{MLR}_\mu$.

The remainder of the article is organized as follows. In Section 2, we provide some background on computability theory and algorithmic randomness. Section 2.5 contains a discussion of the basic properties of complex sequences. The relationship between complex sequences and randomness with respect to computable, continuous measures is investigated in Section 3. In Section 4 we study the behavior of complex proper sequences in the context of atomic measures. Next, in Section 5, we consider non-complex proper sequences. Lastly, in Section 6 we relate the results of Section 5 to the class of diminutive measures.

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