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# Proof lengths for instances of the Paris–Harrington principle

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ABSTRACT

As Paris and Harrington have famously shown, Peano Arithmetic does not prove that for all numbers $k, m, n$ there is an $N$ which satisfies the statement $\mathrm{PH}(k, m, n, N)$: For any $k$-coloring of its $n$-element subsets the set $\{0, \ldots, N-1\}$ has a large homogeneous subset of size $\geq m$. At the same time very weak theories can establish the $\Sigma_1$-statement $\exists_N \mathrm{PH}(\overline{k}, \overline{m}, \overline{n}, N)$ for any fixed parameters $k, m, n$. Which theory, then, does it take to formalize *natural* proofs of these instances? It is known that $\forall_m \exists_N \mathrm{PH}(\overline{k}, m, \overline{n}, N)$ has a natural and short proof (relative to $n$ and $k$) by $\Sigma_{n-1}$-induction. In contrast, we show that there is an elementary function $e$ such that any proof of $\exists_N \mathrm{PH}(\overline{e(n)}, \overline{n+1}, \overline{n}, N)$ by $\Sigma_{n-2}$-induction is ridiculously long. In order to establish this result on proof lengths we give a computational analysis of slow provability, a notion introduced by Sy-David Friedman, Rathjen and Weiermann. We will see that slow uniform $\Sigma_1$-reflection is related to a function that has a considerably lower growth rate than $F_{\varepsilon_0}$ but dominates all functions $F_\alpha$ with $\alpha < \varepsilon_0$ in the fast-growing hierarchy.



We recall some terminology from [17]: For a set $X$ and a natural number $n$ we write $[X]^n$ for the collection of subsets of $X$ with precisely $n$ elements. Given a function $f$ with domain $[X]^n$, a subset $Y$ of $X$ is called homogeneous for $f$ if the restriction of $f$ to the set $[Y]^n$ is constant. A non-empty subset of $\mathbb{N}$ is called large if its cardinality is at least as big as its minimal element. Where the context suggests it we use $N$ to denote the set $\{0, \ldots, N-1\}$. Then the Paris–Harrington Principle, or Strengthened Finite Ramsey Theorem, expresses that for all natural numbers $k, m, n$ there is an $N$ such that the following statement holds:

$$\mathrm{PH}(k, m, n, N) \quad :\equiv \quad \text{“for any function } [N]^n \to k \text{ the set } N \text{ has a large homogeneous subset with at least } m \text{ elements”}$$

Using the methods presented in [8, Section I.1(b)] it is easy to formalize the statement $\mathrm{PH}(k, m, n, N)$ in the language of first order arithmetic, as a formula that is $\Delta_1$ in the theory $\mathbf{I\Sigma}_1$ of $\Sigma_1$-induction. The celebrated result of [17] says that the formula $\forall_{k,m,n} \exists_N \mathrm{PH}(k, m, n, N)$ is true but unprovable in Peano Arithmetic.

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As is well-known, any true $\Sigma_1$-formula in the language of first-order arithmetic can be proved in a theory as weak as Robinson Arithmetic. It is thus pointless to ask whether a $\Sigma_1$-sentence is provable in a sound arithmetical theory, in contrast to the situation for $\Pi_1$-sentences (cf. Gödel's Theorems) and $\Pi_2$-sentences (provably total functions). What we can sensibly ask is whether a $\Sigma_1$-sentence has a proof with some additional property. The present paper explores this question for instances $\exists_N \mathrm{PH}(\overline{k}, \overline{m}, \overline{n}, N)$ of the Paris–Harrington Principle. Our principal result states that, for some elementary function $e$, the following holds:

$$\begin{array}{l}\text{For sufficiently large } n\text{, no proof of the formula } \exists_N \mathrm{PH}(\overline{e(n)}, \overline{n+1}, \overline{n}, N)\\ \text{in the theory } \mathbf{I\Sigma}_{n-2} \text{ can have Gödel number smaller than } F_{\varepsilon_0}(n-3).\end{array} \tag{1}$$

If we replace $\mathbf{I\Sigma}_{n-2}$ by $\mathbf{I\Sigma}_{n-3}$ (and $F_{\varepsilon_0}(n-3)$ by $F_{\varepsilon_0}(n-4)$) then we can take the constant function $e(n) = 8$. It is open whether we can make $e$ constant and keep the stronger fragment $\mathbf{I\Sigma}_{n-2}$.

Recall that $F_{\varepsilon_0}$ is the function at stage $\varepsilon_0$ of the fast-growing hierarchy. Ketonen and Solovay in [11] have related it to the function that maps $(k, m, n)$ to the smallest witness $N$ which makes the statement $\mathrm{PH}(\overline{k}, \overline{m}, \overline{n}, \overline{N})$ true. A classical result due to Kreisel, Wainer and Schwichtenberg [13,19,24] says that $F_{\varepsilon_0}$ eventually dominates any provably total function of Peano Arithmetic. Similar to (1) we will show that the $\Sigma_1$-formula $\exists_y F_{\varepsilon_0}(\overline{n}) = y$ has no short proof in the theory $\mathbf{I\Sigma}_n$.

By [8, Theorem II.1.9] the formula $\forall_m \exists_N \mathrm{PH}(\overline{k}, m, \overline{n}, N)$ is provable in $\mathbf{I\Sigma}_{n-1}$, for each fixed $n \geq 2$ and $k$. The proofs of these instances formalize perfectly natural mathematical arguments. According to [8, Section II.2(c)] they can be constructed in the meta-theory $\mathbf{I\Sigma}_1$. Since all provably total functions of $\mathbf{I\Sigma}_1$ are primitive recursive, this complements (1) by the following statement:

$$\begin{array}{l}\text{There is a primitive recursive function which maps } (k, n) \text{ with } n \geq 2 \text{ to}\\ \text{a proof of the formula } \forall_m \exists_N \mathrm{PH}(\overline{k}, m, \overline{n}, N) \text{ in the theory } \mathbf{I\Sigma}_{n-1}.\end{array} \tag{2}$$

Similarly, a primitive recursive construction yields proofs of $\exists_y F_{\varepsilon_0}(\overline{n}) = y$ in the theories $\mathbf{I\Sigma}_{n+1}$: In view of $F_{\varepsilon_0}(x) \simeq F_{\omega_{x+1}}(x) = F_{\omega_x^{x+1}}(x)$ it suffices to prove the statements "$F_{\omega_n^{n+1}}$ is total". This is done by $\Pi_2$-induction up to $\omega_n^{n+1}$, which is available in $\mathbf{I\Sigma}_{n+1}$ by Gentzen's classical construction (cf. [4, Theorem 4.11]).

We argue that (1) is not only a result about proof length, but also about the existence of natural proofs: Observe first that we are concerned with sequences $p_n$ of proofs for a sequence of parametrized statements $A_n$, rather than with a single proof of a single statement. Under which conditions can such a sequence of proofs follow an intelligible uniform proof idea? It is the role of the proofs $p_n$ to guarantee that the formulas $A_n$ are true. On the other hand the statement "the given proof idea leads to formally correct proofs $p_n$ of the statements $A_n$" should, we believe, be justified by fairly elementary means. Since elementary means cannot prove the totality of functions with a high growth rate this implies that the function mapping $n$ to (a code of) the proof $p_n$ cannot grow too fast. In this sense (1) shows that $\mathbf{I\Sigma}_{n-2}$-proofs of the Paris–Harrington Principle for arity $n$ and $e(n)$ colors cannot follow a natural proof idea. The author sees no formal condition which would, on the positive side, ensure that a sequence of proofs is natural. On an informal level the construction which establishes [8, Theorem II.1.9] appears to provide natural $\mathbf{I\Sigma}_{n-1}$-proofs of the statements $\forall_m \exists_N \mathrm{PH}(\overline{k}, m, \overline{n}, N)$.

Let us briefly discuss connections with a line of research initiated by Harvey Friedman: Theorem 15 in [21] says that any proof of a certain $\Sigma^0_1$-statement in the theory $\Pi^1_2$-$\mathrm{BI}_0$ must have at least $2_{1000}$ (i.e. 1000 iterated exponentials to the base 2) symbols. Obviously this goes much further than our result insofar as it involves a much stronger theory. However, there is also a more conceptual difference: Friedman's statement can, in principle, be verified explicitly (by looking at all possible proofs with less than $2_{1000}$ symbols) and is thus finitistically meaningful. In contrast, our statement (1) involves an unbounded existential quantifier, implicit in the phrase "sufficiently large". It is conceivable that any witness to this existential quantifier is