Annals of Pure and Applied Logic 168 (2017) 1383–1395

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Superstability from categoricity in abstract elementary classes ☆

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A R T I C L E   I N F O

Article history:
Received 27 September 2016
Received in revised form 5 January 2017
Accepted 6 January 2017
Available online 27 March 2017

MSC:
primary 03C48
secondary 03C45, 03C55, 03E05

Keywords:
Abstract elementary classes
Categoricity
Superstability
Splitting
Coheir
Forking

A B S T R A C T

Starting from an abstract elementary class with no maximal models, Shelah and Villaveces have shown (assuming instances of diamond) that categoricity implies a superstability-like property for nonsplitting, a particular notion of independence. We generalize their result as follows: given any abstract notion of independence for Galois (orbital) types over models, we derive that the notion satisfies a superstability property provided that the class is categorical and satisfies a weakening of amalgamation. This extends the Shelah–Villaveces result (the independence notion there was splitting) as well as a result of the first and second author where the independence notion was coheir. The argument is in ZFC and fills a gap in the Shelah–Villaveces proof.

1. Introduction

1.1. General motivation and history

Forking is one of the central notions of model theory, discovered and developed by Shelah in the seventies for stable and NIP theories [13]. One way to extend Shelah's first-order stability theory is to move beyond first-order. In the mid seventies, Shelah did this by starting the program of *classification theory for non-elementary classes* focusing first on classes axiomatizable in $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ [12] and later on the more

☆ This material is based upon work done while the first author was supported by the National Science Foundation under Grant No. DMS-1402191.

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http://dx.doi.org/10.1016/j.apal.2017.01.005
0168-0072/

general abstract elementary classes (AECs) [14]. Roughly, an AEC is a pair $\mathcal{K} = (K, \prec_{\mathcal{K}})$ satisfying some of the basic category-theoretic properties of $(\mathrm{Mod}(T), \prec)$ (but not the compactness theorem). Among the central problems, there are the decades-old categoricity and eventual categoricity conjectures of Shelah. In this paper, we assume that the reader has a basic knowledge of AECs, see for example [4] or [2].

One key shift in this program is the move away from syntactic types (studied in the $\mathbb{L}_{\lambda^+,\omega}$ context by [5,6,11] and others) and towards a semantic notion of type, introduced in [15] and named *Galois type* by Grossberg [4].[1] This has an easy definition when the class $\mathcal{K}$ has amalgamation, joint embedding and no maximal models, as these properties allow us to assume that all the elements of $\mathcal{K}$ we would like to discuss are substructures of a "monster" model $\mathfrak{C} \in \mathcal{K}$. In that case, $\mathrm{gtp}(\mathbf{b}/A)$ is defined as the orbit of $\mathbf{b}$ under the action of the group $\mathrm{Aut}_A(\mathfrak{C})$ on $\mathfrak{C}$. One can also develop the notion of Galois type without the above assumption, however then the definition is more technical.

1.2. Independence, superstability, and no long splitting chains in AECs

In [17] a first candidate for an independence relation was introduced: the notion of μ-splitting (for $M_0 \prec_{\mathcal{K}} M$ both in $\mathcal{K}_\mu$, $p \in \mathrm{gS}(M)$ *μ-splits over* M_0 provided there are $M_0 \prec_{\mathcal{K}} M_\ell \prec_{\mathcal{K}} M$, $\ell = 1, 2$ and $f : M_1 \cong_{M_0} M_2$ such that $f(p \upharpoonright M_1) \neq p \upharpoonright M_2$).

This notion was used by Shelah to establish a downward version of his categoricity conjecture from a successor for classes having the amalgamation property. Later similar arguments [7,8] were used to derive a strong upward version of Shelah's conjecture for classes satisfying the additional locality property of (Galois) types called tameness.

In Chapter II of [18], Shelah introduced good λ-frames: an axiomatic definition of forking on Galois types over models of size λ. The notion is, by definition, required to satisfy basic properties of forking in superstable first-order theories (e.g. symmetry, extension, uniqueness, and local character). The theory of good λ-frames is well-developed and has had several applications to the categoricity conjecture (see Chapters III and IV of [18] and recent work of the fourth author [25–28]).

Constructions of good frames rely on weaker independence notions like nonsplitting, see e.g. [23,24]. A key property of splitting in these constructions is that there is "no long splitting chains in $\mathcal{K}_\mu$": if $\langle M_i : i \leq \alpha \rangle$ is an increasing continuous chain in $\mathcal{K}_\mu$ (so $\alpha < \mu^+$ is a limit ordinal) and M_{i+1} is universal over M_i for each $i < \alpha$, then for any $p \in \mathrm{gS}(M_\alpha)$ there exists $i < \alpha$ so that p does not μ-split over M_i (this is called *strong universal local character at* α in the present paper, see Definition 6). This can be seen as a replacement for the statement "every type does not fork over a finite set". The property is already studied in [17], and has several nontrivial consequences: for example (assuming amalgamation, joint embedding, no maximal models, stability in μ, and tameness), no long splitting chains in $\mathcal{K}_\mu$ implies that $\mathcal{K}$ is stable everywhere above μ [24, Theorem 5.6] and has a good μ^+-frame on the subclass of saturated models of cardinality μ^+ [23, Corollary 6.14]. No long splitting chains has consequences for the uniqueness of limit models, another superstability-like property saying in essence that saturated models can be built in few steps (see for example [19–22]).

The first and second authors have explored another approach to independence by adapting the notion of coheir to AECs. They have shown that for classes satisfying amalgamation which are also tame and short (a strengthening of tameness, using the variables of a type instead of its parameters), failure of a certain order property implies that coheir has some basic properties of forking from a stable first-order theory. There the "no long coheir chain" property also has strong consequences (for example on the uniqueness of limit models [3, Corollary 6.18]).

[1] Shelah uses the name *orbital types* in some later papers.