ARTICLE IN PRESS APAL:2568

Annals of Pure and Applied Logic ••• (••••) •••–•••

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Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

The Gamma question for many-one degrees ☆

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A R T I C L E I N F O

Article history:
Received 9 September 2016
Received in revised form 9 December 2016
Accepted 18 January 2017
Available online xxxx

MSC:
03D32
68Q30
03D30
03D80

Keywords:
Computability theory
Coarse computability
Gamma question

A B S T R A C T

A set A is coarsely computable with density $r \in [0, 1]$ if there is an algorithm for deciding membership in A which always gives a (possibly incorrect) answer, and which gives a correct answer with density at least r. To any Turing degree $\mathbf{a}$ we can assign a value $\Gamma_T(\mathbf{a})$: the minimum, over all sets A in $\mathbf{a}$, of the highest density at which A is coarsely computable. The closer $\Gamma_T(\mathbf{a})$ is to 1, the closer $\mathbf{a}$ is to being computable. Andrews, Cai, Diamondstone, Jockusch, and Lempp noted that Γ_T can take on the values 0, 1/2, and 1, but not any values in strictly between 1/2 and 1. They asked whether the value of Γ_T can be strictly between 0 and 1/2. This is the Gamma question.

Replacing Turing degrees by many-one degrees, we get an analogous question, and the same arguments show that Γ_m can take on the values 0, 1/2, and 1, but not any values strictly between 1/2 and 1. We will show that for any $r \in [0, 1/2]$, there is an m-degree $\mathbf{a}$ with $\Gamma_m(\mathbf{a}) = r$. Thus the range of Γ_m is $[0, 1/2] \cup \{1\}$.

Benoit Monin has recently announced a solution to the Gamma question for Turing degrees. Interestingly, his solution gives the opposite answer: the only possible values of Γ_T are 0, 1/2, and 1.

1. Introduction

We give a solution to the Gamma question for many-one degrees by showing that for each $r \in [0, 1/2]$, there is a many-one degree $\mathbf{a}$ such that $\Gamma_m(\mathbf{a}) = r$.

A set $A \subseteq \omega$ is *coarsely computable* if, roughly speaking, we have an algorithm for deciding membership in A which always gives an answer, and the answer is correct except on a set of density zero. By density, we mean asymptotic lower density.

☆ The author was partially supported by NSERC PGSD3-454386-2014. The author would like to thank Antonio Montalbán, James Walsh, Carl Jockusch, and André Nies for their helpful comments.

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http://dx.doi.org/10.1016/j.apal.2017.01.006
0168-0072/© 2017 Published by Elsevier B.V.

Please cite this article in press as: M. Harrison-Trainor, The Gamma question for many-one degrees, Ann. Pure Appl. Logic (2017), http://dx.doi.org/10.1016/j.apal.2017.01.006

Definition 1. The *lower density* of a set $Z \subseteq \omega$ is

$$\underline{\rho}(Z) := \liminf_{n\to\infty} \frac{|Z \cap [0,n)|}{n}.$$

More generally, we can talk about algorithms which are correct half the time, or a third of the time, or almost never. To a set $A \subseteq \omega$, we can assign a real number which measures the highest density to which it can be approximated by a computable set.

Definition 2 *([4]).* A set $A \subseteq \omega$ is *coarsely computable at density* $r \in [0,1]$ if there is a computable set R such that $\underline{\rho}(A \leftrightarrow R) = r$. Here, $A \leftrightarrow R$ is the set on which A and R agree:

$$A \leftrightarrow R := \{x \mid x \in A \iff x \in R\}.$$

Definition 3 *([4]).* The *coarse computability bound* of a set $A \subseteq \omega$ is

$$\gamma(A) := \sup\{r \mid A \text{ is coarsely computable at density } r\}.$$

That is, $\gamma(A)$ is the supremum, over all computable sets R, of $\underline{\rho}(A \leftrightarrow R)$.

It is known that for each $r \in (0,1]$, there are sets with coarse computability bound r such that the supremum is obtained, and sets where the supremum is not obtained [4].

Jockusch and Schupp [7] have shown that every non-zero Turing degree contains a set which is not coarsely computable. (This follows from the proof of Proposition 6 below.) Thus, if $\Gamma_T(\mathbf{a}) = 1$, then $\mathbf{a} = \mathbf{0}$. Andrews, Cai, Diamondstone, Jockusch, and Lempp suggested assigning to each Turing degree a real number which measures the extent to which all sets computable in that degree can be coarsely computed.

Definition 4 *([1]).* The *coarse computability bound* of a Turing degree $\mathbf{a}$ is

$$\Gamma_T(\mathbf{a}) := \inf\{\gamma(A) \mid A \text{ is } \mathbf{a}\text{-computable}\}.$$

It suffices to take the infimum only over sets in $\mathbf{a}$.

Andrews, Cai, Diamondstone, Jockusch, and Lempp showed that $\Gamma_T(\mathbf{a})$ can take on the values 0, 1/2, and 1.[1]

Theorem 5 *([1]). For a Turing degree* $\mathbf{a}$*:*

(1) *If* $\mathbf{a}$ *is computable,* $\Gamma_T(\mathbf{a}) = 1$.
(2) *If* $\mathbf{a}$ *is computably traceable and non-computable,* $\Gamma_T(\mathbf{a}) = 1/2$.
(3) *If* $\mathbf{a}$ *is 1-random and hyperimmune-free,* $\Gamma_T(\mathbf{a}) = 1/2$.
(4) *If* $\mathbf{a}$ *is hyperimmune,* $\Gamma_T(\mathbf{a}) = 0$.
(5) *If* $\mathbf{a}$ *is PA,* $\Gamma_T(\mathbf{a}) = 0$.

Hirschfeldt, Jockusch, McNicholl, and Schupp showed that $\Gamma_T(\mathbf{a})$ cannot take on any values in the open interval $(1/2, 1)$. We will repeat the proof here because we will reference it later.

Proposition 6 *([4]). Let* $\mathbf{a}$ *be a nonzero Turing degree. Then* $\Gamma_T(\mathbf{a}) \leq \frac{1}{2}$.

[1] See also [8] for a unifying approach to some of these examples.

Please cite this article in press as: M. Harrison-Trainor, The Gamma question for many-one degrees, Ann. Pure Appl. Logic (2017), http://dx.doi.org/10.1016/j.apal.2017.01.006