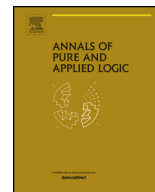




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## The Gamma question for many-one degrees ☆

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## ABSTRACT

A set  $A$  is coarsely computable with density  $r \in [0, 1]$  if there is an algorithm for deciding membership in  $A$  which always gives a (possibly incorrect) answer, and which gives a correct answer with density at least  $r$ . To any Turing degree  $\mathbf{a}$  we can assign a value  $\Gamma_T(\mathbf{a})$ : the minimum, over all sets  $A$  in  $\mathbf{a}$ , of the highest density at which  $A$  is coarsely computable. The closer  $\Gamma_T(\mathbf{a})$  is to 1, the closer  $\mathbf{a}$  is to being computable. Andrews, Cai, Diamondstone, Jockusch, and Lempp noted that  $\Gamma_T$  can take on the values 0,  $1/2$ , and 1, but not any values in strictly between  $1/2$  and 1. They asked whether the value of  $\Gamma_T$  can be strictly between 0 and  $1/2$ . This is the Gamma question.

Replacing Turing degrees by many-one degrees, we get an analogous question, and the same arguments show that  $\Gamma_m$  can take on the values 0,  $1/2$ , and 1, but not any values strictly between  $1/2$  and 1. We will show that for any  $r \in [0, 1/2]$ , there is an  $m$ -degree  $\mathbf{a}$  with  $\Gamma_m(\mathbf{a}) = r$ . Thus the range of  $\Gamma_m$  is  $[0, 1/2] \cup \{1\}$ .

Benoit Monin has recently announced a solution to the Gamma question for Turing degrees. Interestingly, his solution gives the opposite answer: the only possible values of  $\Gamma_T$  are 0,  $1/2$ , and 1.

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## 1. Introduction

We give a solution to the Gamma question for many-one degrees by showing that for each  $r \in [0, 1/2]$ , there is a many-one degree  $\mathbf{a}$  such that  $\Gamma_m(\mathbf{a}) = r$ .

A set  $A \subseteq \omega$  is *coarsely computable* if, roughly speaking, we have an algorithm for deciding membership in  $A$  which always gives an answer, and the answer is correct except on a set of density zero. By density, we mean asymptotic lower density.

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**Definition 1.** The *lower density* of a set  $Z \subseteq \omega$  is

$$\underline{\rho}(Z) := \liminf_{n \rightarrow \infty} \frac{|Z \cap [0, n]|}{n}.$$

More generally, we can talk about algorithms which are correct half the time, or a third of the time, or almost never. To a set  $A \subseteq \omega$ , we can assign a real number which measures the highest density to which it can be approximated by a computable set.

**Definition 2** ([4]). A set  $A \subseteq \omega$  is *coarsely computable at density*  $r \in [0, 1]$  if there is a computable set  $R$  such that  $\underline{\rho}(A \leftrightarrow R) = r$ . Here,  $A \leftrightarrow R$  is the set on which  $A$  and  $R$  agree:

$$A \leftrightarrow R := \{x \mid x \in A \iff x \in R\}.$$

**Definition 3** ([4]). The *coarse computability bound* of a set  $A \subseteq \omega$  is

$$\gamma(A) := \sup\{r \mid A \text{ is coarsely computable at density } r\}.$$

That is,  $\gamma(A)$  is the supremum, over all computable sets  $R$ , of  $\underline{\rho}(A \leftrightarrow R)$ .

It is known that for each  $r \in (0, 1]$ , there are sets with coarse computability bound  $r$  such that the supremum is obtained, and sets where the supremum is not obtained [4].

Jockusch and Schupp [7] have shown that every non-zero Turing degree contains a set which is not coarsely computable. (This follows from the proof of Proposition 6 below.) Thus, if  $\Gamma_T(\mathbf{a}) = 1$ , then  $\mathbf{a} = \mathbf{0}$ . Andrews, Cai, Diamondstone, Jockusch, and Lempp suggested assigning to each Turing degree a real number which measures the extent to which all sets computable in that degree can be coarsely computed.

**Definition 4** ([1]). The *coarse computability bound* of a Turing degree  $\mathbf{a}$  is

$$\Gamma_T(\mathbf{a}) := \inf\{\gamma(A) \mid A \text{ is } \mathbf{a}\text{-computable}\}.$$

It suffices to take the infimum only over sets in  $\mathbf{a}$ .

Andrews, Cai, Diamondstone, Jockusch, and Lempp showed that  $\Gamma_T(\mathbf{a})$  can take on the values 0,  $1/2$ , and 1.<sup>1</sup>

**Theorem 5** ([1]). For a Turing degree  $\mathbf{a}$ :

- (1) If  $\mathbf{a}$  is computable,  $\Gamma_T(\mathbf{a}) = 1$ .
- (2) If  $\mathbf{a}$  is computably traceable and non-computable,  $\Gamma_T(\mathbf{a}) = 1/2$ .
- (3) If  $\mathbf{a}$  is 1-random and hyperimmune-free,  $\Gamma_T(\mathbf{a}) = 1/2$ .
- (4) If  $\mathbf{a}$  is hyperimmune,  $\Gamma_T(\mathbf{a}) = 0$ .
- (5) If  $\mathbf{a}$  is PA,  $\Gamma_T(\mathbf{a}) = 0$ .

Hirschfeldt, Jockusch, McNicholl, and Schupp showed that  $\Gamma_T(\mathbf{a})$  cannot take on any values in the open interval  $(1/2, 1)$ . We will repeat the proof here because we will reference it later.

**Proposition 6** ([4]). Let  $\mathbf{a}$  be a nonzero Turing degree. Then  $\Gamma_T(\mathbf{a}) \leq \frac{1}{2}$ .

<sup>1</sup> See also [8] for a unifying approach to some of these examples.

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