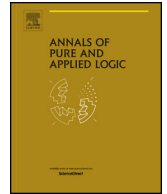




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Reducts of the Henson graphs with a constant

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ABSTRACT

Let (H_n, E) denote the Henson graph, the unique countable homogeneous graph whose age consists of all finite K_n -free graphs. In this note the reducts of the Henson graphs with a constant are determined up to first-order interdefinability. It is shown that up to first-order interdefinability $(H_3, E, 0)$ has 13 reducts and $(H_n, E, 0)$ has 16 reducts for $n \geq 4$.

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1. Introduction

For $n \geq 3$ we denote by (H_n, E) the unique countable homogeneous graph that embeds a finite graph A if and only if A is K_n -free, where K_n denotes the complete graph on n vertices. A countable structure Δ is *homogeneous* if every isomorphism between finite induced substructures extends to an automorphism of Δ . The graphs (H_n, E) were first constructed by C.W. Henson in [7]. A.H. Lachlan and R. Woodrow [10] have shown that apart from trivial examples, the random graph (R, E) , the Henson graphs (H_n, E) and their complements are the only countably infinite homogeneous graphs. By homogeneity, vertices of (H_n, E) are indistinguishable: for all $u, v \in (H_n, E)$, there exists an automorphism $\alpha \in \text{Aut}(H_n, E)$ such that $\alpha(u) = v$. Hence, there is no ambiguity in the notation $(H_n, E, 0)$: it denotes the structure that we obtain by adding

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a constant symbol 0 to the signature of (H_n, E) and interpret it as a vertex of (H_n, E) . In this paper, we classify the structures that are first-order definable (without parameters) in $(H_n, E, 0)$, i.e., the *reducts* of $(H_n, E, 0)$.

The first result of this form is due to P.J. Cameron [6], who has shown that the dense linear order $(\mathbb{Q}, <)$ has five reducts up to *first-order interdefinability*. Two structures Γ and Δ are first-order interdefinable if Γ has a first-order definition (without parameters) in Δ and vice versa, i.e., if they are reducts of one another. S. Thomas [13] proved that the random graph (R, E) has five reducts up to first-order interdefinability, and determined the reducts of the random k -uniform hypergraph for all $k \geq 2$ in [14]. In [13] it was shown that the Henson graphs (H_n, E) have no proper non-trivial reducts, i.e.,

Theorem 1.1. [Thomas] *Every reduct of (H_n, E) is first-order interdefinable either with (H_n, E) itself or with $(H_n, =)$ for all $n \geq 3$.*

In [13] Thomas posed the following conjecture.

Conjecture 1. *Every countable homogeneous structure over a finite relational language has finitely many reducts up to first-order interdefinability.*

J.H. Bennett has shown that the conjecture holds for the random tournament in [1]. Recently, M. Junker and M. Ziegler [9] proved that $(\mathbb{Q}, <, 0)$ has 116 reducts up to first-order interdefinability.

The purpose of this paper is to verify Thomas' conjecture for $(H_n, E, 0)$ for all $n \geq 3$. Note that $(H_n, E, 0)$ is indeed first-order interdefinable with a structure that is homogeneous in a finite *relational* language (see Remark 2.1). There is an essential difference between the result for $n = 3$ and for $n \geq 4$. Up to first-order interdefinability $(H_3, E, 0)$ has 13 reducts, and $(H_n, E, 0)$ has 16 reducts for $n \geq 4$ (see Theorem 2.3). This characterisation is based on the Nešetřil–Rödl theorem in [11] and a method introduced by M. Bodirsky and M. Pinsker applied in [2–5]. The current note is the first implementation of the Bodirsky–Pinsker method to obtain a new first-order characterisation of the reducts of a homogeneous structure.

2. The main result

2.1. Closed groups

Let D be a countable set. A relational structure $\Gamma = (D, (Q_j)_{j \in J})$ is a *reduct* of $\Delta = (D, (R_i)_{i \in I})$ if Q_j is first-order definable from the set of relations $\{R_i \mid i \in I\}$ for all $j \in J$. If Γ is a reduct of Δ , then clearly $\text{Aut}(\Delta) \subseteq \text{Aut}(\Gamma)$. If Δ is ω -categorical, then the converse also holds by (a consequence of) Ryll–Nardzewski's theorem (see in [8]). A countable structure is ω -categorical if it is the unique countable model of its first-order theory up to isomorphism. If Δ is a countable structure that is homogeneous in a finite relational language, then Δ is ω -categorical, thus Ryll–Nardzewski's theorem [8] establishes a Galois connection between reducts of Δ and subgroups of $\text{Sym}(D)$ that contain $\text{Aut}(\Delta)$. Throughout the paper, $\text{Sym}(D)$ denotes the full symmetric group acting on D , i.e., the group of all permutations of D . This Galois connection is given by the operators Aut mapping reducts to their automorphism groups, and Inv mapping permutation groups $\text{Aut}(\Delta) \subseteq G \subseteq \text{Sym}(D)$ to the structure with all relations on D that are invariant under the action of G . Just like every Galois connection, this gives rise to a closure operator. In our case, a permutation group $\text{Aut}(\Delta) \subseteq G \subseteq \text{Sym}(D)$ is *closed* if $G = \text{Aut}(\Gamma)$ for some reduct Γ of Δ . Equivalently, G is closed if whenever $\alpha \in \text{Sym}(D)$ is such that for all finite $F \subseteq D$ there exists a $\gamma \in G$ with $\alpha \upharpoonright_F = \gamma \upharpoonright_F$, then $\alpha \in G$. Moreover, given two reducts Γ_1 and Γ_2 of Δ , Γ_1 is a reduct of Γ_2 if and only if $\text{Aut}(\Gamma_2) \subseteq \text{Aut}(\Gamma_1)$. In particular, Γ_1 and Γ_2 are first-order interdefinable if and only if $\text{Aut}(\Gamma_1) = \text{Aut}(\Gamma_2)$. Thus reducts of a countable, homogeneous structure Δ in a finite relational language up to first-order interdefinability can

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