ARTICLE IN PRESS
APAL:2571

Annals of Pure and Applied Logic ••• (••••) •••–•••

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Reducts of the Henson graphs with a constant

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ARTICLE INFO

Article history:
Received 6 March 2012
Received in revised form 27 June 2015
Accepted 29 June 2015
Available online xxxx

MSC:
03C10
03C35
03C40
03C50

Keywords:
First-order
Reduct
Ramsey
Homogeneous
Automorphism

ABSTRACT

Let (H_n, E) denote the Henson graph, the unique countable homogeneous graph whose age consists of all finite K_n-free graphs. In this note the reducts of the Henson graphs with a constant are determined up to first-order interdefinability. It is shown that up to first-order interdefinability $(H_3, E, 0)$ has 13 reducts and $(H_n, E, 0)$ has 16 reducts for $n \geq 4$.

1. Introduction

For $n \geq 3$ we denote by (H_n, E) the unique countable homogeneous graph that embeds a finite graph A if and only if A is K_n-free, where K_n denotes the complete graph on n vertices. A countable structure Δ is *homogeneous* if every isomorphism between finite induced substructures extends to an automorphism of Δ. The graphs (H_n, E) were first constructed by C.W. Henson in [7]. A.H. Lachlan and R. Woodrow [10] have shown that apart from trivial examples, the random graph (R, E), the Henson graphs (H_n, E) and their complements are the only countably infinite homogeneous graphs. By homogeneity, vertices of (H_n, E) are indistinguishable: for all $u, v \in (H_n, E)$, there exists an automorphism $\alpha \in \mathrm{Aut}(H_n, E)$ such that $\alpha(u) = v$. Hence, there is no ambiguity in the notation $(H_n, E, 0)$: it denotes the structure that we obtain by adding

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[1] The author has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007–2013 Grant Agreement No. 257039).

http://dx.doi.org/10.1016/j.apal.2017.01.009
0168-0072/

Please cite this article in press as: A. Pongrácz, Reducts of the Henson graphs with a constant, Ann. Pure Appl. Logic (2017), http://dx.doi.org/10.1016/j.apal.2017.01.009

a constant symbol 0 to the signature of (H_n, E) and interpret it as a vertex of (H_n, E). In this paper, we classify the structures that are first-order definable (without parameters) in $(H_n, E, 0)$, i.e., the *reducts* of $(H_n, E, 0)$.

The first result of this form is due to P.J. Cameron [6], who has shown that the dense linear order $(\mathbb{Q}, <)$ has five reducts up to *first-order interdefinability*. Two structures Γ and Δ are first-order interdefinable if Γ has a first-order definition (without parameters) in Δ and vice versa, i.e., if they are reducts of one another. S. Thomas [13] proved that the random graph (R, E) has five reducts up to first-order interdefinability, and determined the reducts of the random k-uniform hypergraph for all $k \geq 2$ in [14]. In [13] it was shown that the Henson graphs (H_n, E) have no proper non-trivial reducts, i.e.,

Theorem 1.1. *[Thomas] Every reduct of (H_n, E) is first-order interdefinable either with (H_n, E) itself or with $(H_n, =)$ for all $n \geq 3$.*

In [13] Thomas posed the following conjecture.

Conjecture 1. *Every countable homogeneous structure over a finite relational language has finitely many reducts up to first-order interdefinability.*

J.H. Bennett has shown that the conjecture holds for the random tournament in [1]. Recently, M. Junker and M. Ziegler [9] proved that $(\mathbb{Q}, <, 0)$ has 116 reducts up to first-order interdefinability.

The purpose of this paper is to verify Thomas' conjecture for $(H_n, E, 0)$ for all $n \geq 3$. Note that $(H_n, E, 0)$ is indeed first-order interdefinable with a structure that is homogeneous in a finite *relational* language (see Remark 2.1). There is an essential difference between the result for $n = 3$ and for $n \geq 4$. Up to first-order interdefinability $(H_3, E, 0)$ has 13 reducts, and $(H_n, E, 0)$ has 16 reducts for $n \geq 4$ (see Theorem 2.3). This characterisation is based on the Nešetřil–Rödl theorem in [11] and a method introduced by M. Bodirsky and M. Pinsker applied in [2–5]. The current note is the first implementation of the Bodirsky–Pinsker method to obtain a new first-order characterisation of the reducts of a homogeneous structure.

2. The main result

2.1. Closed groups

Let D be a countable set. A relational structure $\Gamma = (D, (Q_j)_{j\in J})$ is a *reduct* of $\Delta = (D, (R_i)_{i\in I})$ if Q_j is first-order definable from the set of relations $\{R_i \mid i \in I\}$ for all $j \in J$. If Γ is a reduct of Δ, then clearly $\mathrm{Aut}(\Delta) \subseteq \mathrm{Aut}(\Gamma)$. If Δ is ω-categorical, then the converse also holds by (a consequence of) Ryll–Nardzewski's theorem (see in [8]). A countable structure is *ω-categorical* if it is the unique countable model of its first-order theory up to isomorphism. If Δ is a countable structure that is homogeneous in a finite relational language, then Δ is ω-categorical, thus Ryll–Nardzewski's theorem [8] establishes a Galois connection between reducts of Δ and subgroups of $\mathrm{Sym}(D)$ that contain $\mathrm{Aut}(\Delta)$. Throughout the paper, $\mathrm{Sym}(D)$ denotes the full symmetric group acting on D, i.e., the group of all permutations of D. This Galois connection is given by the operators Aut mapping reducts to their automorphism groups, and Inv mapping permutation groups $\mathrm{Aut}(\Delta) \subseteq G \subseteq \mathrm{Sym}(D)$ to the structure with all relations on D that are invariant under the action of G. Just like every Galois connection, this gives rise to a closure operator. In our case, a permutation group $\mathrm{Aut}(\Delta) \subseteq G \subseteq \mathrm{Sym}(D)$ is *closed* if $G = \mathrm{Aut}(\Gamma)$ for some reduct Γ of Δ. Equivalently, G is closed if whenever $\alpha \in \mathrm{Sym}(D)$ is such that for all finite $F \subseteq D$ there exists a $\gamma \in G$ with $\alpha\restriction_F = \gamma\restriction_F$, then $\alpha \in G$. Moreover, given two reducts Γ_1 and Γ_2 of Δ, Γ_1 is a reduct of Γ_2 if and only if $\mathrm{Aut}(\Gamma_2) \subseteq \mathrm{Aut}(\Gamma_1)$. In particular, Γ_1 and Γ_2 are first-order interdefinable if and only if $\mathrm{Aut}(\Gamma_1) = \mathrm{Aut}(\Gamma_2)$. Thus reducts of a countable, homogeneous structure Δ in a finite relational language up to first-order interdefinability can