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Distance structures for generalized metric spaces

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ABSTRACT

Let $\mathcal{R} = (R, \oplus, \leq, 0)$ be an algebraic structure, where $\oplus$ is a commutative binary operation with identity 0, and $\leq$ is a translation-invariant total order with least element 0. Given a distinguished subset $S \subseteq R$, we define the natural notion of a "generalized" $\mathcal{R}$-metric space, with distances in S. We study such metric spaces as first-order structures in a relational language consisting of a distance inequality for each element of S. We first construct an ordered additive structure $\mathcal{S}^*$ on the space of quantifier-free 2-types consistent with the axioms of $\mathcal{R}$-metric spaces with distances in S, and show that, if A is an $\mathcal{R}$-metric space with distances in S, then any model of $\mathrm{Th}(A)$ logically inherits a canonical $\mathcal{S}^*$-metric. Our primary application of this framework concerns countable, universal, and homogeneous metric spaces, obtained as generalizations of the rational Urysohn space. We adapt previous work of Delhommé, Laflamme, Pouzet, and Sauer to fully characterize the existence of such spaces. We then fix a countable totally ordered commutative monoid $\mathcal{R}$, with least element 0, and consider $\mathcal{U}_\mathcal{R}$, the countable Urysohn space over $\mathcal{R}$. We show that quantifier elimination for $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is characterized by continuity of addition in $\mathcal{R}^*$, which can be expressed as a first-order sentence of $\mathcal{R}$ in the language of ordered monoids. Finally, we analyze an example of Casanovas and Wagner in this context.

The fundamental objects of interest in this paper are metric spaces. Specifically, we study the behavior of metric spaces as combinatorial structures in relational languages. This is the setting of a vast body of literature (e.g. [7,11,21,25–27]) focusing on topological dynamics of automorphism groups and Ramsey properties of countable homogeneous structures. Our goal is to develop the model theory of metric spaces in this setting. We face the immediate obstacle that the notion of "metric space" is not very well controlled by classical first-order logic, in the sense that models of the theory of a metric space need not be metric spaces. Indeed, this is a major motivation for working in continuous logic and model theory for *metric structures*, which are always complete metric spaces with the metric built into the logic (see [4]). However, we wish to study the model theory of (possibly incomplete) metric spaces treated as combinatorial structures (specifically, labeled graphs where complexity is governed by the triangle inequality). In some sense, we will

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sacrifice the global topological structure of metric spaces for the sake of understanding local combinatorial complexity. We will also develop an algebraic structure on distances sets of metric spaces, as a means to analyze this combinatorial complexity.

Another benefit of our framework is that it will be flexible enough to encompass generalized metric spaces with distances in arbitrary ordered additive structures. This setting appears often in the literature, with an obvious example of extracting a metric from a valuation. Other examples include [19], where Narens considers topological spaces "metrizable" by a generalized metric over an ordered abelian group, as well as [18], where Morgan and Shalen use metric spaces over ordered abelian groups to generalize the notion of an $\mathbb{R}$-tree. Also, in [7], Casanovas and Wagner use the phenomenon of "infinitesimal distance" to construct a theory without the strict order property that does not eliminate hyperimaginaries. We will analyze this example at the end of Section 7.

We will consider metric spaces as first-order *relational* structures. However, when working outside of this first-order setting, it will usually be much more convenient to think of metric spaces as "sorted" structures consisting of a set of points together with a distance function into a set of distances. Distinguishing between these two viewpoints will be especially important, and so we will very carefully explain the precise first-order relational setting in which we will be working. This explanation requires the following basic definitions.

Definition 0.1. Let $\mathcal{L}_{\mathrm{om}} = \{\oplus, \leq, 0\}$ be the **language of ordered monoids** consisting of a binary function symbol $\oplus$, a binary relation symbol $\leq$, and a constant symbol 0. Fix an $\mathcal{L}_{\mathrm{om}}$-structure $\mathcal{R} = (R, \oplus, \leq, 0)$.

1. $\mathcal{R}$ is a **distance magma** if
   - (i) (*totality*) $\leq$ is a total order on R;
   - (ii) (*positivity*) $r \leq r \oplus s$ for all $r, s \in R$;
   - (iii) (*order*) for all $r, s, t, u \in R$, if $r \leq t$ and $s \leq u$ then $r \oplus s \leq t \oplus u$;
   - (iv) (*commutativity*) $r \oplus s = s \oplus r$ for all $r, s \in R$;
   - (v) (*unity*) $r \oplus 0 = r = 0 \oplus r$ for all $r \in R$.
2. $\mathcal{R}$ is a **distance monoid** if it is a distance magma and
   - (vi) (*associativity*) $(r \oplus s) \oplus t = r \oplus (s \oplus t)$ for all $r, s, t \in R$.

Note that if $\mathcal{R}$ is a distance magma, then it follows from the *positivity* and *unity* axioms that 0 is the least element of R. Moreover, given $r, s, t \in R$ if $r \leq s$ then $r \oplus t \leq s \oplus t$ by the *order* axiom. However, it is worth emphasizing that this translation-invariance is not strict: we may have $r < s$, while $r \oplus t = s \oplus t$. In particular, a distance magma may be finite, in which case if $s \in R$ is the maximal element then $r \oplus s = s$ for all $r \in R$. See Example 0.4 below.

Remark 0.2. Recall that, according to [6], a *magma* is simply a set together with a binary operation. After consulting standard literature on ordered algebraic structures (e.g. [8]), one might refer to a distance magma as a *totally and positively ordered commutative unital magma*, and a distance monoid as a *totally and positively ordered commutative monoid*. So our terminology is partly chosen for the sake of brevity. We are separating the associativity axiom because it is not required for our initial results and, more importantly, associativity will frequently characterize some useful combinatorial property of metric spaces (see Proposition 4.9(e), Proposition 5.7, Exercise 5.11).

Next, we observe that the notion of a distance magma allows for a reasonable definition of a generalized metric space. Definitions of a similar flavor can be found in [1,18], and [19].

Definition 0.3. Suppose $\mathcal{R} = (R, \oplus, \leq, 0)$ is a distance magma. Fix a nonempty set A and a function $d : A \times A \longrightarrow R$. We call (A, d) an **$\mathcal{R}$-colored space**, and define the **distance set of** (A, d), denoted $\mathrm{Dist}(A, d)$, to be the image of d in R. Given an $\mathcal{R}$-colored space (A, d), we say d is an **$\mathcal{R}$-metric on** A if