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Downward categoricity from a successor inside a good frame ☆

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ABSTRACT

In the setting of abstract elementary classes (AECs) with amalgamation, Shelah has proven a downward categoricity transfer from categoricity in a successor and Grossberg and VanDieren have established an upward transfer assuming in addition a locality property for Galois types that they called tameness.

We further investigate categoricity transfers in tame AECs. We use orthogonality calculus to prove a downward transfer from categoricity in a successor in AECs that have a good frame (a forking-like notion for types of singletons) on an interval of cardinals:

Theorem 0.1. *Let* $\mathbf{K}$ *be an AEC and let* $LS(\mathbf{K}) \leq \lambda < \theta$ *be cardinals. If* $\mathbf{K}$ *has a type-full good* $[\lambda, \theta]$*-frame and* $\mathbf{K}$ *is categorical in both* λ *and* θ^+*, then* $\mathbf{K}$ *is categorical in all* $\mu \in [\lambda, \theta]$.

We deduce improvements on the threshold of several categoricity transfers that do not mention frames. For example, the threshold in Shelah's transfer can be improved from $\beth_{\beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}}$ to $\beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}$ assuming that the AEC is $\mathrm{LS}(\mathbf{K})$-tame. The successor hypothesis can also be removed from Shelah's result by assuming in addition either that the AEC has primes over sets of the form $M \cup \{a\}$ or (using an unpublished claim of Shelah) that the weak generalized continuum hypothesis holds.

1. Introduction

1.1. Motivation and history

In his two volume book [32,33] on classification theory for abstract elementary classes (AECs), Shelah introduces the notion of a *good* λ*-frame* [32, II.2.1]. Roughly, a good λ-frame is a local notion of independence for types of length one over models of size λ. The independence notion satisfies basic properties of forking

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in a superstable first-order theory. Good frames are the central concept of the book. In Chapters II and III, Shelah discusses the following three questions regarding frames:

Question 1.1.

(1) Given an AEC $\mathbf{K}$, when does there exist a good λ-frame $\mathfrak{s}$ whose underlying AEC $\mathbf{K}_{\mathfrak{s}}$ is $\mathbf{K}_\lambda$ (or some subclass of saturated models in $\mathbf{K}_\lambda$)?
(2) Given a good λ-frame, under what conditions can it be extended to a good λ^+-frame?
(3) Once one has a good frame, how can one prove categoricity transfers?

Shelah's answers (see for example II.3.7, III.1, and III.2 in [32]) involve a mix of set-theoretic hypotheses (such as the weak generalized continuum hypothesis: $2^\theta < 2^{\theta^+}$ for all cardinals θ) and strong local model-theoretic hypotheses (such as few models in λ^{++}). While Shelah's approach is very powerful (for example in [32, Chapter IV], Shelah proves the eventual categoricity conjecture in AECs with amalgamation assuming some set-theoretic hypotheses, see more below), most of his results do not hold in ZFC.

An alternate approach is to make *global* model-theoretic assumptions. In [17], Grossberg and VanDieren introduced *tameness*, a locality property which says that Galois types are determined by their small restrictions. In [9], Boney showed that in an AEC which is λ-tame for types of length two and has amalgamation, a good λ-frame can be extended to all models of size at least λ (we call the resulting object a good $(\geq \lambda)$-frame, and similarly define good $[\lambda, \theta]$-frame for $\theta > \lambda$ a cardinal). In [13], tameness for types of length two was improved to tameness for types of length one. In particular, the answer to Question 1.1.(2) is always positive in tame AECs with amalgamation. As for existence (Question 1.1.(1)), we showed in [44] how to build good frames in tame AECs with amalgamation assuming categoricity in a cardinal of high-enough cofinality. Further improvements were made in [11,43,46]. This gives answers to Questions 1.1.(1), (2) in tame AECs with amalgamation:

Fact 1.2. Let $\mathbf{K}$ be an AEC with amalgamation and let $\lambda \geq \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is λ-tame.

(1) [13, Corollary 6.9] If there is a good λ-frame $\mathfrak{s}$ with $\mathbf{K}_{\mathfrak{s}} = \mathbf{K}_\lambda$, then $\mathfrak{s}$ can be extended to a good $(\geq \lambda)$-frame (with underlying class $\mathbf{K}$).
(2) [46, Corollary 6.14] If $\mathbf{K}$ has no maximal models and is categorical in some $\mu > \lambda$, then there is a type-full good λ^+-frame with underlying class the Galois saturated models of $\mathbf{K}$ of size λ^+.

1.2. Categoricity in good frames

In this paper, we study Question 1.1.(3) in the global setting: assuming the existence of a good frame together with some global model-theoretic properties, what can we say about the categoricity spectrum? From the two results above, it is natural to assume that we are already working inside a type-full good $(\geq \lambda)$-frame (this implies properties such as λ-tameness and amalgamation). It is then known how to transfer categoricity with the additional assumption that the class has primes over sets of the form $M \cup \{a\}$. This has been used to prove Shelah's eventual categoricity conjecture for universal classes [41,42].

Definition 1.3 *(III.3.2 in [32]).* An AEC $\mathbf{K}$ *has primes* if for any nonalgebraic Galois type $p \in \mathrm{gS}(M)$ there exists a triple (a, M, N) such that $p = \mathrm{gtp}(a/M; N)$ and for every $N' \in \mathbf{K}$, $a' \in |N'|$, such that $p = \mathrm{gtp}(a'/M; N')$, there exists $f : N \xrightarrow[M]{} N'$ with $f(a) = a'$.