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Moduli space of cubic Newton maps

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A B S T R A C T

In this article, we study the topology and bifurcations of the moduli space $\mathcal{M}_3$ of cubic Newton maps. It's a subspace of the moduli space of cubic rational maps, carrying the Riemann orbifold structure $(\widehat{\mathbb{C}}, (2, 3, \infty))$. We prove two results:

• The boundary of the unique unbounded hyperbolic component is a Jordan arc and the boundaries of all other hyperbolic components are Jordan curves;
• The Head's angle map is surjective and monotone. The fibers of this map are characterized completely.

The first result is a moduli space analogue of the first author's dynamical regularity theorem [37]. The second result confirms a conjecture of Tan Lei.

1. Introduction

Let P be a polynomial of degree $d \geq 2$. It can be written as

$$P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0,$$

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where $a_0, \cdots, a_d$ are complex numbers and $a_d \neq 0$. The *Newton's method* N_P of P is defined by

$$N_P(z) = z - \frac{P(z)}{P'(z)}.$$

The method, also known as the *Newton–Raphson method* named after Isaac Newton and Joseph Raphson, was first proposed to find successively better approximations to the roots (or zeros) of a real-valued function. In 1879, Arthur Cayley [6] first noticed the difficulties in generalizing the Newton's method to complex roots of polynomials with degree greater than 2 and complex initial values. This opened the way to study the theory of iterations of holomorphic functions, as initiated by Pierre Fatou and Gaston Julia around 1920. In the literature, N_P is also called the *Newton map* of P. The study of Newton maps attracts a lot of people both in complex dynamical systems and in computational mathematics.

1.1. What is known

The Newton maps can be viewed as a dynamical system as well as a root-finding algorithm. Therefore, it provides a rich source to study from various purposes. Here is an incomplete list of what's known for Newton maps from different views:

Topology of Julia set: The simple connectivity of the immediate attracting basins of cubic Newton maps was first proven by Przytycki [29]. Shishikura [40] proved that the Julia sets of the Newton maps of polynomials are always connected by means of quasiconformal surgery. Applying the Yoccoz puzzle theory, Roesch [37] proved the local connectivity of the Julia sets for most cubic Newton maps.

The combinatorial structure of the Julia sets of cubic Newton maps was first studied by Janet Head [12]. With the help of Thurston's theory on characterization of rational maps, Tan Lei [42] showed that every post-critically finite cubic Newton map can be constructed by mating two cubic polynomials; Building on the thesis [26], Lodge, Mikulich and Schleicher [18,19] gave a combinatorial classification of post-critically finite Newton maps.

Root-finding algorithm: As a root-finding algorithm, Newton's method is effective for quadratic polynomials but may fail in the cubic case. McMullen [24] exhibited a generally convergent algorithm (apparently different from Newton's method) for cubics and proved that there are no generally convergent purely iterative algorithms for solving polynomials of degrees four or more. On the other hand, by generalizing a previous result of Manning [22], Hubbard, Schleicher and Sutherland [15] proved that for every $d \geq 2$, there is a finite universal set S_d with cardinality at most $O(d \log^2 d)$ such that for any root of any suitably normalized polynomial of degree d, there is an initial point in S_d whose orbit converges to this root under iterations of its Newton map. For further extensions of these results, see [39] and the references therein.