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Necessary conditions and nonexistence results for connected submanifolds in a Riemannian manifold

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ABSTRACT

In this paper, we derive density estimates for submanifolds with variable mean curvature in a Riemannian manifold with sectional curvature bounded above by a constant. This leads to distance estimates for the boundaries of compact connected submanifolds. As applications, we give several necessary conditions and nonexistence results for compact connected minimal submanifolds, Bryant surfaces, and surfaces with small L^2 norm of the mean curvature vector in a Riemannian manifold.

1. Introduction

Douglas [8] and Radó [18] gave the first solution to the Plateau problem independently, which says that any simple closed curve in $\mathbb{R}^3$ bounds at least one minimal disk. In order

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to generalize the original Plateau problem, one may ask whether any given two disjoint simple closed curves Γ_1 and Γ_2 in $\mathbb{R}^3$ bound a compact connected minimal surface or not. Douglas [9] showed that if D_1 and D_2 are the least area minimal disks bounded by Γ_1 and Γ_2 respectively, and if the following condition (so-called Douglas condition) is satisfied:

$$\inf\{\mathrm{Area}(S)\} < \mathrm{Area}(D_1) + \mathrm{Area}(D_2),$$

then there exists a minimal annulus bounded by Γ_1 and Γ_2. Here the infimum is taken over all surfaces of annular type spanning Γ_1 and Γ_2. However, the solution to this generalized Plateau problem does not exist in general. For instance, consider a pair of coaxial circles of fixed radii lying in parallel planes in $\mathbb{R}^3$. It is well-known that if the distance of the planes is sufficiently small, then there exists a minimal annulus spanning two circles. Indeed, it is a part of a catenoid. It is obvious that if the planes are far apart, then there does no longer exist a catenoid bounded by the given pair of circles. Therefore it is interesting to give a quantitative description for the necessary condition on the boundary of compact connected minimal surfaces.

In his series of papers [13–16], Nitsche obtained necessary conditions that two disjoint simple closed curves Γ_1 and Γ_2 lying in parallel planes in $\mathbb{R}^3$ bound a minimal annulus. In fact, he showed that if the diameter of Γ_i is d_i for $i = 1, 2$, and the distance of the two planes is r, then

$$r \leq \frac{3}{2}\max(d_1, d_2).$$

In other words, if the distance between two planes is bigger than $\frac{3}{2}\max(d_1, d_2)$, then there is no minimal annulus bounded by $\Gamma_1 \cup \Gamma_2$. More generally, consider higher-dimensional minimal submanifolds in Euclidean space. Almgren [1] obtained the existence of a number $r > 0$ such that, for two disjoint $(n-1)$-dimensional compact submanifolds Γ_1 and Γ_2 in $\mathbb{R}^{n+k}$, if $\mathrm{dist}(\Gamma_1, \Gamma_2) > r$, then there does not exist n-dimensional compact connected minimal submanifold in $\mathbb{R}^{n+k}$ bounded by $\Gamma_1 \cup \Gamma_2$.

Hildebrandt [11] was able to generalize the Nitsche's result by using the maximum principle for subharmonic functions on the minimal surface in $\mathbb{R}^3$. His idea is based on the following observations:

(1) The coordinate functions x, y, z are harmonic on a minimal surface $\Sigma \subset \mathbb{R}^3$, and hence the quadratic polynomial

$$Q(x, y, z) := x^2 + y^2 - z^2$$

is subharmonic on Σ.

(2) The convex-hull property of a compact minimal surface shows that if two simple closed curves Γ_1 and Γ_2 are separated by the two disjoint components of the solid