

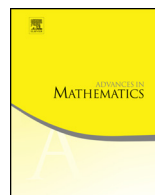


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# A transversal of full outer measure

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## ABSTRACT

We show that for every partition of a set of reals into countable sets there is a transversal of the same outer measure.

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## 1. Introduction

Our aim is to prove the following.

**Theorem 1.1.** *Suppose  $\langle X_\alpha : \alpha \in S \rangle$  is a partition of  $X \subseteq [0, 1]$  into countable sets. Then there exists  $Y \subseteq X$  such that  $|Y \cap X_\alpha| = 1$  for each  $\alpha \in S$  and  $\mu^*(Y) = \mu^*(X)$ .*

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Here  $\mu^*$  denotes Lebesgue outer measure on  $\mathbb{R}$ . For partitions into finite sets, this follows from an old result of Lusin [9] which says that any set of reals can be partitioned into two sets of full outer measure (see Lemma 2.2). Another special case of the above theorem was established in [8]: Every set of reals has a subset of full outer measure that avoids rational distances. The proof given there relied on a theorem of Gitik and Shelah [4–6] which says that forcing with a sigma ideal cannot be isomorphic to a product of random and Cohen forcing (we give another proof of this in Theorem A.1). As a byproduct of our proof, we get a generalization of this theorem to a larger class of forcings (see Lemma 6.1) – For example, an  $\omega$ -length finite support iteration of random forcing. For background on forcing and generic ultrapowers, we refer the reader to [2,7].

**On notation:** For a set of reals  $X$ , by  $\text{env}(X)$  (envelope of  $X$ ), we mean a  $G_\delta$  set  $G$  containing  $X$  such that  $G \setminus X$  has zero inner measure. All relations involving envelopes are supposed to hold modulo null sets. A subset  $Y$  of  $X$  has full outer measure in  $X$  if  $\text{env}(X) = \text{env}(Y)$ . If  $Y \subseteq X$  and  $\text{env}(X) \neq \text{env}(X \setminus Y)$  we say that  $Y$  has positive inner measure in  $X$ ; otherwise, we say that  $Y$  has zero inner measure in  $X$ . For  $T \subseteq {}^{<\omega}2$ , define  $[T] = \{x \in 2^\omega : (\forall n < \omega)(x \upharpoonright n \in T)\}$ . For  $\sigma \in {}^{<\omega}2$ , define  $[\sigma] = \{x \in 2^\omega : \sigma \preceq x\}$ . In forcing, we use the convention that a larger condition is the stronger one – So  $p \leq q$  means  $q$  extends  $p$ . If  $\mathbb{Q}, \mathbb{P}$  are forcing notions, we write  $\mathbb{Q} \subseteq \mathbb{P}$  if  $\mathbb{Q} \subseteq \mathbb{P}$  and every maximal antichain in  $\mathbb{Q}$  is also a maximal antichain in  $\mathbb{P}$ . For an ideal  $\mathcal{I}$  over a set  $X$ , define the following.

- $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$ ;
- $\text{add}(\mathcal{I})$  is the least cardinal  $\kappa$  satisfying: there exists  $\mathcal{F} \subseteq \mathcal{I}$ ,  $|\mathcal{F}| \leq \kappa$  and  $\bigcup \mathcal{F} \notin \mathcal{I}$ ;
- For  $Y \in \mathcal{I}^+$ ,  $\mathcal{I} \upharpoonright Y = \{W \subseteq Y : W \in \mathcal{I}\}$  is the restriction of  $\mathcal{I}$  to  $Y$ .

**2. A sufficient condition**

Without loss of generality,  $(\forall \alpha \in S)(|X_\alpha| = \aleph_0)$ . For each  $\alpha \in S$ , let  $X_\alpha = \{x_{\alpha,n} : n < \omega\}$ . Put  $Y_n = \{x_{\alpha,n} : \alpha \in S\}$ . For  $W \subseteq S$ , write  $Y_n \upharpoonright W = \{x_{\alpha,n} : \alpha \in W\}$ . Note that  $Y_n$  depends on the specific enumeration of  $X_\alpha$  we fixed.

**Claim 2.1.** *It is enough to show the following.*

( $\star$ ): *For every  $X \subseteq [0, 1]$ , for every partition  $\langle X_\alpha : \alpha \in S \rangle$  of  $X$  into  $\aleph_0$ -sized subsets, for every enumeration  $X_\alpha = \{x_{\alpha,n} : n < \omega\}$  (so we can speak of  $Y_n$ 's w.r.t. this enumeration), there is a subset  $W$  of  $S$  such that either*

- (a)  $Y_0$  is null or
- (b)  $Y_0 \upharpoonright W$  has positive outer measure and for all  $n \geq 1$ ,  $Y_n \upharpoonright W$  has zero inner measure in  $Y_n$ .

**Proof of Claim 2.1.** Assume ( $\star$ ). It is enough to show that we can strengthen “ $Y_0 \upharpoonright W$  has positive outer measure” to “ $\mu^*(Y_0 \upharpoonright W) \geq 0.5(\mu^*(Y_0))$ ” in ( $\star$ ) above. For then we can inductively construct a sequence  $\langle (W_i, n_i) : i < \omega \rangle$  such that

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