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A transversal of full outer measure

Ashutosh Kumar [a,*,1], Saharon Shelah [a,b,2]

[a] *Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Edmond J Safra Campus, Givat Ram, Jerusalem 91904, Israel*
[b] *Department of Mathematics, Rutgers, The State University of New Jersey, Hill Center-Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA*

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ABSTRACT

We show that for every partition of a set of reals into countable sets there is a transversal of the same outer measure.
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1. Introduction

Our aim is to prove the following.

Theorem 1.1. *Suppose $\langle X_\alpha : \alpha \in S\rangle$ is a partition of $X \subseteq [0,1]$ into countable sets. Then there exists $Y \subseteq X$ such that $|Y \cap X_\alpha| = 1$ for each $\alpha \in S$ and $\mu^\star(Y) = \mu^\star(X)$.*

* Corresponding author.
E-mail addresses: akumar@math.huji.ac.il (A. Kumar), shelah@math.huji.ac.il (S. Shelah).
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Here $\mu^\star$ denotes Lebesgue outer measure on $\mathbb{R}$. For partitions into finite sets, this follows from an old result of Lusin [9] which says that any set of reals can be partitioned into two sets of full outer measure (see Lemma 2.2). Another special case of the above theorem was established in [8]: Every set of reals has a subset of full outer measure that avoids rational distances. The proof given there relied on a theorem of Gitik and Shelah [4–6] which says that forcing with a sigma ideal cannot be isomorphic to a product of random and Cohen forcing (we give another proof of this in Theorem A.1). As a byproduct of our proof, we get a generalization of this theorem to a larger class of forcings (see Lemma 6.1) – For example, an ω-length finite support iteration of random forcing. For background on forcing and generic ultrapowers, we refer the reader to [2,7].

On notation: For a set of reals X, by $\mathrm{env}(X)$ (envelope of X), we mean a G_δ set G containing X such that $G \setminus X$ has zero inner measure. All relations involving envelopes are supposed to hold modulo null sets. A subset Y of X has full outer measure in X if $\mathrm{env}(X) = \mathrm{env}(Y)$. If $Y \subseteq X$ and $\mathrm{env}(X) \neq \mathrm{env}(X \setminus Y)$ we say that Y has positive inner measure in X; otherwise, we say that Y has zero inner measure in X. For $T \subseteq {}^{<\omega}2$, define $[T] = \{x \in 2^\omega : (\forall n < \omega)(x \restriction n \in T)\}$. For $\sigma \in {}^{<\omega}2$, define $[\sigma] = \{x \in 2^\omega : \sigma \preceq x\}$. In forcing, we use the convention that a larger condition is the stronger one – So $p \leq q$ means q extends p. If $\mathbb{Q}, \mathbb{P}$ are forcing notions, we write $\mathbb{Q} \lessdot \mathbb{P}$ if $\mathbb{Q} \subseteq \mathbb{P}$ and every maximal antichain in $\mathbb{Q}$ is also a maximal antichain in $\mathbb{P}$. For an ideal $\mathcal{I}$ over a set X, define the following.

- $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$;
- $\mathsf{add}(\mathcal{I})$ is the least cardinal κ satisfying: there exists $\mathcal{F} \subseteq \mathcal{I}$, $|\mathcal{F}| \leq \kappa$ and $\bigcup \mathcal{F} \notin \mathcal{I}$;
- For $Y \in \mathcal{I}^+$, $\mathcal{I} \restriction Y = \{W \subseteq Y : W \in \mathcal{I}\}$ is the restriction of $\mathcal{I}$ to Y.

2. A sufficient condition

Without loss of generality, $(\forall \alpha \in S)(|X_\alpha| = \aleph_0)$. For each $\alpha \in S$, let $X_\alpha = \{x_{\alpha,n} : n < \omega\}$. Put $Y_n = \{x_{\alpha,n} : \alpha \in S\}$. For $W \subseteq S$, write $Y_n \restriction W = \{x_{\alpha,n} : \alpha \in W\}$. Note that Y_n depends on the specific enumeration of X_α we fixed.

Claim 2.1. *It is enough to show the following.*

($\star$): *For every $X \subseteq [0,1]$, for every partition $\langle X_\alpha : \alpha \in S\rangle$ of X into $\aleph_0$-sized subsets, for every enumeration $X_\alpha = \{x_{\alpha,n} : n < \omega\}$ (so we can speak of Y_n's w.r.t. this enumeration), there is a subset W of S such that either*

(a) *Y_0 is null or*
(b) *$Y_0 \restriction W$ has positive outer measure and for all $n \geq 1$, $Y_n \restriction W$ has zero inner measure in Y_n.*

Proof of Claim 2.1. Assume ($\star$). It is enough to show that we can strengthen "$Y_0 \restriction W$ has positive outer measure" to "$\mu^\star(Y_0 \restriction W) \geq 0.5(\mu^\star(Y_0))$" in ($\star$) above. For then we can inductively construct a sequence $\langle (W_i, n_i) : i < \omega\rangle$ such that

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