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A self-similar measure with dense rotations, singular projections and discrete slices ☆

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ABSTRACT

We construct a planar homogeneous self-similar measure, with strong separation, dense rotations and dimension greater than 1, such that there exist lines for which dimension conservation does not hold and the projection of the measure is singular. In fact, the set of such directions is residual and the typical slices of the measure, perpendicular to these directions, are discrete.

1. Introduction and statement of results

Let R be a 2×2 rotation matrix, with $R^n \neq Id$ for all $n \geq 1$, and let $r \in (0, 1)$. Consider a homogeneous IFS on $\mathbb{R}^2$

$$\{\varphi_i(x) = rRx + a_i\}_{i \in I},$$

with the strong separation condition (SSC), and a self-similar measure

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$$\mu = \sum_{i \in I} p_i \cdot \varphi_i \mu \,.$$

It is among the most basic planar self-similar measures. Hence it is a natural question in fractal geometry to study the dimension and continuity of the projections $\{P_u\mu\}_{u\in S}$ and slices

$$\{\{\mu_{u,x}\}_{x\in\mathbb{R}^2} \,:\, u \in S\}\,.$$

Here S is the unit circle of $\mathbb{R}^2$, P_u is the orthogonal projection onto the line spanned by u, and $\{\mu_{u,x}\}_{x\in\mathbb{R}^2}$ is the disintegration of μ with respect to $P_u^{-1}(\mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}^2$. A more elaborate description of these disintegrations is given in Section 2. Note that the atoms of $P_u^{-1}(\mathcal{B})$ are lines perpendicular to $\operatorname{span}\{u\}$.

Dimensionwise, the behaviour of the projections is as regular as possible. Indeed, Hochman and Shmerkin [9] have proven that $P_u\mu$ is exact dimensional, with

$$\dim P_u\mu = \min\{1, \dim\mu\},$$

for each $u \in S$. A version of this, for self-similar sets with dense rotations, was first proven by Peres and Shmerkin [16]. Considering the continuity of the projections, Shmerkin and Solomyak [18] have shown, assuming $\dim\mu > 1$, that the set

$$E = \{u \in S \,:\, P_u\mu \text{ is singular}\}$$

has zero Hausdorff dimension.

Let us turn to discuss the concept of dimension conservation and the dimension of slices. A Borel probability measure ν on $\mathbb{R}^2$ is said to be dimension conserving (DC), with respect to the projection P_u, if

$$\dim_H \nu = \dim_H P_u\nu + \dim_H \nu_{u,x} \text{ for } \nu\text{-a.e. } x \in \mathbb{R}^2,$$

where $\dim_H$ stands for Hausdorff dimension. It always holds that ν is DC with respect to P_u for almost every $u \in S$. This follows from results, valid for general measures, regarding the typical dimension of projections (see [10]) and slices (see [12]). Falconer and Jin [5] have shown that ν is DC, with respect to P_u for all $u \in S$, whenever ν is self-similar with a finite rotation group. An analogous statement, for self-similar sets with the SSC, was first proven by Furstenberg [8]. Another related result for sets is due to Falconer and Jin [6]. They showed that if $K \subset \mathbb{R}^2$ is self-similar, with $\dim K > 1$ and a dense rotation group, then for every $\epsilon > 0$ there exists $N_\epsilon \subset S$, with $\dim_H N_\epsilon = 0$, such that for $u \in S \setminus N_\epsilon$ the set

$$\{x \in \operatorname{span}\{u\} \,:\, \dim_H(K \cap P_u^{-1}\{x\}) > \dim K - 1 - \epsilon\}$$

has positive length.