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Euler class groups and the homology of elementary and special linear groups

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ABSTRACT

We improve homology stability ranges for elementary and special linear groups over rings with many units. Our result implies stability for unstable Quillen K-groups and proves a conjecture of Bass. For commutative local rings with infinite residue fields, we show that the obstruction to further stability is given by Milnor–Witt K-theory. As an application we construct Euler classes of projective modules with values in the cohomology of the Milnor–Witt K-theory sheaf. For d-dimensional commutative noetherian rings with infinite residue fields we show that the vanishing of the Euler class is necessary and sufficient for an oriented projective module P of rank d to split off a rank 1 free direct summand. Along the way we obtain a new presentation of Milnor–Witt K-theory and of symplectic K_2 simplifying the classical Matsumoto–Moore presentation.

Contents

1. Introduction 2
2. The homology of affine groups 6
3. Stability in homology and K-theory 12
4. Milnor–Witt K-theory 21

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5. The obstruction to further stability 29
5.1. Multiplicative properties of the spectral sequence 29
5.2. Presentation and decomposability 35
5.3. The Steinberg relation and $H_2(SL_2A)$ 43
5.4. Centrality of $[-1,1]$ and $H_n(SL_nA, SL_{n-1}A)$ 50
5.5. Prestability 56
6. Euler class groups 59
Appendix A. The affine B.G.-property for the Zariski topology 69
References 80

1. Introduction

The purpose of this paper is to improve stability ranges in homology and algebraic K-theory of elementary and special linear groups, and to apply these results to construct obstruction classes for projective modules to split off a free direct summand.

Our first result concerns a conjecture of Bass [5, Conjecture XVI on p. 43]. In [5] he conjectured that for a commutative noetherian ring A whose maximal ideal spectrum has dimension d the canonical maps

$$\pi_i BGL^+_{n-1}(A) \to \pi_i BGL^+_n(A)$$

are surjective for $n \geq d+i+1$ and bijective for $n \geq d+i+2$. Here, for a connected space X, we denote by X^+ Quillen's plus-construction with respect to the maximal perfect subgroup of $\pi_1 X$, and we write $BGL^+_n(A)$ for $BGL_n(A)^+$. In this generality, there are counterexamples to Bass' conjecture; see [35, §8]. The best general positive results to date concerning the conjecture are due to van der Kallen [37] and Suslin [34]. They prove that the maps are surjective for $n-1 \geq \max(2i, \mathrm{sr}(A)+i-1)$ and bijective for $n-1 \geq \max(2i, \mathrm{sr}(A)+i)$ where $\mathrm{sr}(A)$ denotes the stable rank of A [39]. Here A need not be commutative nor noetherian.

In this paper we prove Bass' conjecture for rings with many units. Recall [28] that a ring A (always associative with unit) has many units if for every integer $n \geq 1$ there is a family of n *central elements* of A such that the sum of each non-empty subfamily is a unit. Examples of rings with many units are infinite fields, commutative local rings with infinite residue field and algebras over a ring with many units. Here is our first main result.

Theorem 1.1 *(Theorem 3.10). Let A be a ring with many units. Then the natural homomorphism*

$$\pi_i BGL^+_{n-1}(A) \to \pi_i BGL^+_n(A)$$

is an isomorphism for $n \geq i+\mathrm{sr}(A)+1$ and surjective for $n \geq i+\mathrm{sr}(A)$.