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Sheets and associated varieties of affine vertex algebras

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We show that sheet closures appear as associated varieties of affine vertex algebras. Further, we give new examples of non-admissible affine vertex algebras whose associated variety is contained in the nilpotent cone. We also prove some conjectures from our previous paper and give new examples of lisse affine W-algebras.

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1. Introduction

It is known [47] that every vertex algebra V is canonically filtered and therefore it can be considered as a quantization of its associated graded Poisson vertex algebra $\operatorname{gr} V$. The generating subring R_V of $\operatorname{gr} V$ is called the *Zhu's C_2-algebra* of V [56] and has the structure of a Poisson algebra. Its spectrum

$$\tilde{X}_V = \operatorname{Spec} R_V$$

is called the *associated scheme* of V and the corresponding reduced scheme $X_V = \operatorname{Specm} R_V$ is called the *associated variety* of V ([6,10]). Since it is Poisson, the coordinate ring of its arc space $J_\infty \tilde{X}_V$ has a natural structure of a Poisson vertex algebra ([6]), and there is a natural surjective homomorphism $\mathbb{C}[J_\infty \tilde{X}_V] \to \operatorname{gr} V$, which is in many cases an isomorphism. We have [6] that $\dim \operatorname{Spec}(\operatorname{gr} V) = 0$ if and only if $\dim X_V = 0$, and in this case V is called *lisse* or *C_2-cofinite*.

Recently, associated varieties of vertex algebras have caught attention of physicists since it turned out that the associated variety of a vertex algebra coming [14] from a *four dimensional* $N = 2$ superconformal field theory should coincide with the *Higgs branch* of the corresponding four dimensional theory ([53]).

In the case that V is the simple affine vertex algebra $V_k(\mathfrak{g})$ associated with a finite-dimensional simple Lie algebra $\mathfrak{g}$ at level $k \in \mathbb{C}$, X_V is a Poisson subscheme of $\mathfrak{g}^*$ which is G-invariant and conic, where G is the adjoint group of $\mathfrak{g}$. Note that on the contrary to the associated variety of a primitive ideal of $U(\mathfrak{g})$, the variety $X_{V_k(\mathfrak{g})}$ is not necessarily contained in the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$. In fact, $X_{V_k(\mathfrak{g})} = \mathfrak{g}^*$ for a generic k. On the other hand, $X_{V_k(\mathfrak{g})} = \{0\}$ if and only if $V_k(\mathfrak{g})$ is integrable, that is, k is a non-negative integer. Except for a few cases, the description of X_V is fairly open even for $V = V_k(\mathfrak{g})$, despite of its connection with four dimensional superconformal field theories.

In [7], the first named author showed that $X_{V_k(\mathfrak{g})}$ is the closure of some nilpotent orbit of $\mathfrak{g}^*$ in the case that $V_k(\mathfrak{g})$ is *admissible* [42].

In the previous article [12], we showed that $X_{V_k(\mathfrak{g})}$ is the minimal nilpotent orbit closure in the case that $\mathfrak{g}$ belongs to the Deligne exceptional series [24] and $k = -h^\vee/6-1$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Note that the level $k = -h^\vee/6 - 1$ is not admissible for the types D_4, E_6, E_7, E_8.

In all the above cases, $X_{V_k(\mathfrak{g})}$ is a closure of a nilpotent orbit $\mathbb{O} \subset \mathcal{N}$, or $X_{V_k(\mathfrak{g})} = \mathfrak{g}^*$. Therefore it is natural to ask the following.

Question 1. Are there cases when $X_{V_k(\mathfrak{g})} \not\subset \mathcal{N}$ and $X_{V_k(\mathfrak{g})}$ is a proper subvariety of $\mathfrak{g}^*$? For example, are there cases when $X_{V_k(\mathfrak{g})}$ is the closure of a non-nilpotent *Jordan class* (cf. §2)?

Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through a non-degenerate bilinear form of $\mathfrak{g}$.

Given $m \in \mathbb{N}$, let $\mathfrak{g}^{(m)}$ be the set of elements $x \in \mathfrak{g}$ such that $\dim \mathfrak{g}^x = m$, with $\mathfrak{g}^x$ the centralizer of x in $\mathfrak{g}$. A subset $\mathbb{S} \subset \mathfrak{g}$ is called a *sheet* of $\mathfrak{g}$ if it is an irreducible