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# Finite versus infinite: An insufficient shift

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A B S T R A C T

The shift graph $\mathcal{G}_S$ is defined on the space of infinite subsets of natural numbers by letting two sets be adjacent if one can be obtained from the other by removing its least element. We show that this graph is not a minimum among the graphs of the form $\mathcal{G}_f$ defined on some Polish space $X$, where two distinct points are adjacent if one can be obtained from the other by a given Borel function $f : X \to X$. This answers the primary outstanding question from [8].


A *directed graph* is a pair $\mathcal{G} = (X, R)$ where $R$ is an irreflexive binary relation on $X$. A homomorphism from $\mathcal{G} = (X, R)$ to $\mathcal{G}' = (X', R')$ is a map $h : X \to X'$ such that $(x, y) \in R$ implies $(h(x), h(y)) \in R'$ for all $x, y \in X$. A *coloring* of $\mathcal{G}$ is a map $c : X \to Y$ such that $(x_1, x_2) \in R$ implies $c(x_1) \neq c(x_2)$ for all $(x_1, x_2) \in X \times X$. In case $X$ is a topological space, the *Borel chromatic number* $\chi_B(\mathcal{G})$ of $\mathcal{G}$ is defined by

$$\chi_B(\mathcal{G}) = \min\{|c(X)| \mid c : X \to Y \text{ is a Borel coloring of } \mathcal{G} \text{ in a Polish space } Y\},$$

where $|c(X)|$ denotes the cardinality of the range of $c$.

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In this note we only deal with graphs generated by a function. Let $X$ be a Polish space and $f : X \to X$ is a Borel map. We let $\mathcal{D}_f = (X, D_f)$ be the directed graph given by

$$x \, D_f \, y \quad \longleftrightarrow \quad x \neq y \wedge f(x) = y.$$

We also consider its symmetric counterpart $\mathcal{G}_f = (X, R_f)$ given by

$$x \, R_f \, y \quad \longleftrightarrow \quad x \neq y \wedge (f(x) = y \vee f(y) = x).$$

Notice that clearly $\chi_B(\mathcal{G}_f) = \chi_B(\mathcal{D}_f)$.

The following example has drawn considerable attention in the study of Borel chromatic numbers [2–5]. Let $X = [\omega]^\infty$ be the set of infinite sets of natural numbers with topology induced from the Cantor space $2^\omega$ when $Y \subseteq \omega$ is identified with its characteristic function $\chi_Y : \omega \to 2$. The shift operation $\mathsf{S} : [\omega]^\infty \to [\omega]^\infty$ is the continuous map defined by $\mathsf{S}(Y) = Y \setminus \{\min Y\}$. While $\mathcal{G}_\mathsf{S}$ is an acyclic graph and has therefore chromatic number 2, it follows from the Galvin–Prikry Theorem [6] that $\chi_B(\mathcal{G}_\mathsf{S}) = \aleph_0$.

Kechris, Solecki and Todorčević [8, Problem 8.1] (see also [5, Section 3] and [12]) asked whether the following is true: If $X$ is a Polish space and $f : X \to X$ is a Borel function, then exactly one of the following holds:

1. The Borel chromatic number of $\mathcal{G}_f$ is finite;
2. There is a continuous homomorphism from $\mathcal{G}_\mathsf{S}$ to $\mathcal{G}_f$.

We show that the answer is negative, namely:

**Theorem 1.** *There exists a Polish space $X$ together with a continuous finite-to-1 function $f : X \to X$ such that $\chi_B(\mathcal{G}_f) = \aleph_0$ and there is no Borel homomorphism from $\mathcal{G}_\mathsf{S}$ to $\mathcal{G}_f$.*

We do not have any explicit example witnessing the above existential statement. This is because our proof consists of showing that a certain subset of the set of graphs with the above property is a true $\mathbf{\Pi}^1_2$ set in some suitable standard Borel space.

We can however be a bit more specific. If $P$ is a binary relation on $\omega$, let $\vec{P}$ be the closed subset of $[\omega]^\infty$ defined by

$$\vec{P} = \big\{ (n_i)_{i \in \omega} \in [\omega]^\infty \bigm| \forall i \in \omega \; n_i \, P \, n_{i+1} \big\},$$

where an element of $[\omega]^\infty$ is identified with the enumeration $(n_i)_{i \in \omega}$ of its elements in strictly increasing order. If $\mathcal{G} = (X, R)$ is a directed graph and $Y \subseteq X$ let us denote by $\mathcal{G}|Y$ the restriction of $\mathcal{G}$ to $Y$ given by $(Y, R \cap (Y \times Y))$.

The proof of Theorem 1 actually yields the following result.

**Scholium 2.** *There exists a binary relation $P$ on $\omega$ such that $\chi_B(\mathcal{G}_\mathsf{S}|\vec{P}) = \aleph_0$ and there is no Borel homomorphism from $\mathcal{G}_\mathsf{S}$ to $\mathcal{G}_\mathsf{S}|\vec{P}$.*