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Duality structures and discrete conformal variations of piecewise constant curvature surfaces

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ABSTRACT

A piecewise constant curvature manifold is a triangulated manifold that is assigned a geometry by specifying lengths of edges and stipulating the simplex has an isometric embedding into a constant curvature background geometry (Euclidean, hyperbolic, or spherical) with the specified edge lengths. Additional geometric structure leads to a notion of discrete conformal structure, generalizing circle packings and their generalizations as studied by Thurston and others. We analyze discrete conformal variations of piecewise constant curvature 2-manifolds, giving particular attention to the variation of angles. Formulas are derived for the derivatives of angles in each background geometry, which yield formulas for the derivatives of curvatures and to curvature functionals. Finally, we provide a complete classification of possible definitions of discrete conformal structures in each of the background geometries.

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1. Introduction

A triangulation of a manifold can be given a geometric structure by assigning compatible geometric structures to its component simplices. One of the easiest ways of doing this is to assign constant curvature geometries to the simplices, as these simplices are uniquely determined by their edge lengths. Such a structure gives a finitely parametrized set of geometric structures on a closed manifold.

In Thurston's formulation of the discrete Riemann mapping problem (see [42]) as well as in applied methods such as discrete exterior calculus (see, e.g., [16], [15]), it is important to not only have a piecewise constant curvature metric assigned to simplices, but also to give a structure to the Poincaré dual of the triangulation. Such structures arise naturally as incircle duals in Thurston's formulation of circle packings and as circumcentric duals in discrete exterior calculus. For piecewise Euclidean surfaces and 3-manifolds, in [22] and [24] the first author gives an axiomatic treatment of geometric duality structures that have orthogonal intersections with the primal simplices, and also relates these to discrete conformal variations. The particular type of duality structures in this paper are generalized centers of simplices and other cells formed by these centers.

The goals of the present work is to make precise the parametrization of these geometric structures by partial edge lengths (giving a discrete analogue of a Riemannian metric), define the general form of discrete conformal structures based on an axiomatic development related to conformal variation of angle, and derive a local classification of such structures. The relationship between centers and partial edge lengths requires a definition of extended background geometry, which is essentially the space of possible geometric centers once the triangle is embedded in the geometry. The axiomatic development of conformal structure follows that in [24] for piecewise Euclidean surfaces, while the construction in piecewise hyperbolic and spherical surfaces is new. The general formulas for angle and curvature variation of piecewise hyperbolic and spherical surfaces are new (however, compare [48]), generalizing circle packings and other discrete conformal structures previously studied by many authors (see Section 1.5 for details). The local classification of discrete conformal structures, giving explicit formulas for the structures, is new for each geometry including Euclidean.

We will begin by making these geometric structures precise, and then give precise statements of the main results.

1.1. Geometric structures on triangulations

In this section, we make precise some geometric structures.

Definition 1. A triangulated surface (M,T) is a topological 2-manifold M together with a triangulation T of M. A (triangulated) *piecewise constant curvature surface* (M,T,ℓ) *with background geometry* $\mathbb{G}$ is a triangulated surface (M,T) with a function ℓ on the