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Solvability of Dirac type equations ☆

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ABSTRACT

This paper develops a weighted L^2-method for the (half) Dirac equation. For Dirac bundles over closed Riemann surfaces, we give a sufficient condition for the solvability of the (half) Dirac equation in terms of a curvature integral. Applying this to the Dolbeault–Dirac operator, we establish an automatic transversality criteria for holomorphic curves in Kähler manifolds. On compact Riemannian manifolds, we give a new perspective on some well-known results about the first eigenvalue of the Dirac operator, and improve the estimates when the Dirac bundle has a $\mathbb{Z}_2$-grading. On Riemannian manifolds with cylindrical ends, we obtain solvability in the L^2-spaces with suitable exponential weights while allowing mild negativity of the curvature.

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1. Introduction

In many geometric problems, it is important to determine the solvability of the linear equation

$$Du = f \tag{1}$$

where D is the Dirac operator on some Dirac bundle. For example, the fundamental Dirac operator on spin manifolds ([1]), the Dolbeault–Dirac operator in Kähler geometry, and the twisted Dirac operator in the normal bundle of instantons (associative submanifolds) in G_2 manifolds ([23]). In general, it is not easy to know when (1) is solvable. For the Dirac operator on spin manifolds, a sufficient condition was given by the positivity of the scalar curvature, dating back to a theorem of Lichnerowicz. However, the positive scalar curvature condition is not always necessary, as the Dirac operator on spin manifolds has the remarkable conformal covariance property ([13]), and a conformal change of metric could make the scalar curvature negative somewhere.

In this paper, starting with the Bochner formula, we establish weighted L^2-estimates and existence theorems for the Dirac equation, just as Hörmander's weighted L^2-method for the $\bar{\partial}$-equation ([15], [16]). In applications of the L^2-method, it is very important to construct good weight functions from geometric conditions (e.g. [4], [24–26]). Sometimes one can gain "extra positivity" in suitable weighted L^2-spaces to establish vanishing theorems.

Let $\lambda_{\mathbb{S}}$ be the function on M defined in (18), which pointwisely is the first eigenvalue of some curvature operator. For Dirac equations on 2-dimensional Riemannian manifolds, taking $n = 2$ in Proposition 2.8, we have

Theorem 1.1. *Let $\mathbb{S}$ be a Dirac bundle over a 2-dimensional Riemannian manifold (M, g) and D be the Dirac operator. Suppose there exists a C^2 function $\varphi : M \to \mathbb{R}$ such that $\Delta\varphi + 2\lambda_{\mathbb{S}} \geq 0$ on M. For each $f \in L^2_\varphi(\Omega, \mathbb{S})$ with $\int_M \frac{|f|^2}{\Delta\varphi+2\lambda_{\mathbb{S}}} e^{-\varphi} < \infty$, there exists a section u of $\mathbb{S}$ satisfying*

$$Du = f, \text{ and } \int\limits_M |u|^2 e^{-\varphi} \leq \int\limits_M \frac{|f|^2}{\Delta\varphi + 2\lambda_{\mathbb{S}}} e^{-\varphi}. \tag{2}$$

Our Theorem 1.1 leads to the following solvability criterion of the half Dirac equation on $\mathbb{Z}_2$-graded Dirac bundles (see Section 2.1 for definitions).

Corollary 1.2. *Let $\mathbb{S}$ be a $\mathbb{Z}_2$-graded Dirac bundle over a closed 2-dimensional Riemannian manifold M and $D^\pm$ be the half Dirac operators, then*

$$\lambda_{\min}(D^\pm D^\mp) \geq \frac{2}{\mathrm{Vol}(M)} \int\limits_M \lambda_{\mathbb{S}^\mp}, \tag{3}$$

where $\lambda_{\min}(D^\pm D^\mp)$ is the first eigenvalue of $D^\pm D^\mp$.