

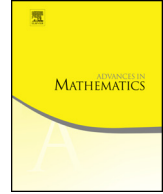


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# Convex hulls of random walks: Expected number of faces and face probabilities <sup>☆</sup>



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## ABSTRACT

Consider a sequence of partial sums  $S_i = \xi_1 + \dots + \xi_i$ ,  $1 \leq i \leq n$ , starting at  $S_0 = 0$ , whose increments  $\xi_1, \dots, \xi_n$  are random vectors in  $\mathbb{R}^d$ ,  $d \leq n$ . We are interested in the properties of the convex hull  $C_n := \text{Conv}(S_0, S_1, \dots, S_n)$ . Assuming that the tuple  $(\xi_1, \dots, \xi_n)$  is exchangeable and a certain general position condition holds, we prove that the expected number of  $k$ -dimensional faces of  $C_n$  is given by the formula

$$\mathbb{E}[f_k(C_n)] = \frac{2 \cdot k!}{n!} \sum_{l=0}^{\infty} \binom{n+1}{d-2l} \left\{ \begin{matrix} d-2l \\ k+1 \end{matrix} \right\},$$

for all  $0 \leq k \leq d-1$ , where  $\binom{n}{m}$  and  $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$  are Stirling numbers of the first and second kind, respectively.

Further, we compute explicitly the probability that for given indices  $0 \leq i_1 < \dots < i_{k+1} \leq n$ , the points  $S_{i_1}, \dots, S_{i_{k+1}}$  form a  $k$ -dimensional face of  $\text{Conv}(S_0, S_1, \dots, S_n)$ . This is

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done in two different settings: for random walks with symmetrically exchangeable increments and for random bridges with exchangeable increments. These results generalize the classical one-dimensional discrete arcsine law for the position of the maximum due to E. Sparre Andersen. All our formulae are distribution-free, that is do not depend on the distribution of the increments  $\xi_k$ 's.

The main ingredient in the proof is the computation of the probability that the origin is absorbed by a *joint* convex hull of several random walks and bridges whose increments are invariant with respect to the action of direct product of finitely many reflection groups of types  $A_{n-1}$  and  $B_n$ . This probability, in turn, is related to the number of Weyl chambers of a product-type reflection group that are intersected by a linear subspace in general position.

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## 1. Statement of main results

### 1.1. Introduction

Let  $\xi_1, \dots, \xi_n$  be (possibly dependent) random  $d$ -dimensional vectors with partial sums

$$S_i = \xi_1 + \dots + \xi_i, \quad 1 \leq i \leq n, \quad S_0 = 0.$$

The sequence  $S_0, S_1, \dots, S_n$  will be referred to as *random walk* or, if the additional boundary condition  $S_n = 0$  is imposed, a *random bridge*.

In the one-dimensional case  $d = 1$ , Sparre Andersen [23–25] derived remarkable formulae for several functionals of the random walk  $S_0, S_1, \dots, S_n$  including the number of positive terms and the position of the maximum. More specifically, assuming that the joint distribution of the increments  $(\xi_1, \dots, \xi_n)$  is invariant under arbitrary permutations and sign changes and that  $\mathbb{P}[S_i = 0] = 0$  for all  $1 \leq i \leq n$ , Sparre Andersen proved in [25, Theorem C] the following *discrete arcsine law* for the position of the maximum:

$$\mathbb{P}[\max\{S_0, \dots, S_n\} = S_i] = \frac{1}{2^{2n}} \binom{2i}{i} \binom{2n-2i}{n-i}, \quad i = 0, \dots, n. \tag{1}$$

By the symmetry, the same holds for the position of the minimum. Surprisingly, the above formula is distribution-free, that is its right-hand side does not depend on the distribution of  $(\xi_1, \dots, \xi_n)$  provided the symmetric exchangeability and the general position assumptions mentioned above are satisfied. Another unexpected consequence of this formula is that the maximum is more likely to be attained at  $i = 0$  or  $i = n$  rather than at  $i \approx n/2$ , as one could naïvely guess. A discussion of the arcsine laws can be found in Feller's book [6, Vol. II, Section XII.8].

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