Advances in Mathematics 320 (2017) 595–629

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Convex hulls of random walks: Expected number of faces and face probabilities ☆

Zakhar Kabluchko [a], Vladislav Vysotsky [b,c], Dmitry Zaporozhets [c]

[a] Institut für Mathematische Stochastik, Westfälische Wilhelms-Universität Münster, Orléans-Ring 10, 48149 Münster, Germany

[b] University of Sussex, Pevensey 2 Building, Falmer Campus, Brighton BN1 9QH, United Kingdom

[c] St. Petersburg Department of Steklov Mathematical Institute, Fontanka 27, 191011 St. Petersburg, Russia

A R T I C L E   I N F O

Article history:
Received 25 February 2017
Received in revised form 21 August 2017
Accepted 23 August 2017
Available online 12 September 2017
Communicated by Erwin Lutwak

MSC:
primary 52A22, 60D05, 60G50
secondary 60G09, 52C35, 20F55, 52B11, 60G70

Keywords:
Convex hull of random walk
Random polytope
Absorption probability
Exchangeability
Hyperplane arrangement
Weyl chamber

A B S T R A C T

Consider a sequence of partial sums $S_i = \xi_1 + \cdots + \xi_i$, $1 \le i \le n$, starting at $S_0 = 0$, whose increments $\xi_1, \ldots, \xi_n$ are random vectors in $\mathbb{R}^d$, $d \le n$. We are interested in the properties of the convex hull $C_n := \mathrm{Conv}(S_0, S_1, \ldots, S_n)$. Assuming that the tuple $(\xi_1, \ldots, \xi_n)$ is exchangeable and a certain general position condition holds, we prove that the expected number of k-dimensional faces of C_n is given by the formula

$$\mathbb{E}[f_k(C_n)] = \frac{2 \cdot k!}{n!} \sum_{l=0}^{\infty} \begin{bmatrix} n+1 \\ d-2l \end{bmatrix} \begin{Bmatrix} d-2l \\ k+1 \end{Bmatrix},$$

for all $0 \le k \le d-1$, where $\left[{n \atop m}\right]$ and $\left\{{n \atop m}\right\}$ are Stirling numbers of the first and second kind, respectively.

Further, we compute explicitly the probability that for given indices $0 \le i_1 < \cdots < i_{k+1} \le n$, the points $S_{i_1}, \ldots, S_{i_{k+1}}$ form a k-dimensional face of $\mathrm{Conv}(S_0, S_1, \ldots, S_n)$. This is

☆ This paper was written when V.V. was affiliated to Imperial College London, where his work was supported by People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement n° [628803]. His work is also supported in part by Grant 16-01-00367 by RFBR. The work of D.Z. is supported in parts by Grant 16-01-00367 by RFBR, the Program of Fundamental Researches of Russian Academy of Sciences "Modern Problems of Fundamental Mathematics", and by Project SFB 1283 of Bielefeld University.

E-mail addresses: zakhar.kabluchko@uni-muenster.de (Z. Kabluchko), v.vysotskiy@sussex.ac.uk, vysotsky@pdmi.ras.ru (V. Vysotsky), zap1979@gmail.com (D. Zaporozhets).

http://dx.doi.org/10.1016/j.aim.2017.09.002

done in two different settings: for random walks with symmetrically exchangeable increments and for random bridges with exchangeable increments. These results generalize the classical one-dimensional discrete arcsine law for the position of the maximum due to E. Sparre Andersen. All our formulae are distribution-free, that is do not depend on the distribution of the increments ξ_k's.
The main ingredient in the proof is the computation of the probability that the origin is absorbed by a *joint* convex hull of several random walks and bridges whose increments are invariant with respect to the action of direct product of finitely many reflection groups of types A_{n-1} and B_n. This probability, in turn, is related to the number of Weyl chambers of a product-type reflection group that are intersected by a linear subspace in general position.

1. Statement of main results

1.1. Introduction

Let $\xi_1, \dots, \xi_n$ be (possibly dependent) random d-dimensional vectors with partial sums

$$S_i = \xi_1 + \dots + \xi_i, \quad 1 \le i \le n, \quad S_0 = 0.$$

The sequence $S_0, S_1, \dots, S_n$ will be referred to as *random walk* or, if the additional boundary condition $S_n = 0$ is imposed, a *random bridge*.

In the one-dimensional case $d = 1$, Sparre Andersen [23–25] derived remarkable formulae for several functionals of the random walk $S_0, S_1, \dots, S_n$ including the number of positive terms and the position of the maximum. More specifically, assuming that the joint distribution of the increments $(\xi_1, \dots, \xi_n)$ is invariant under arbitrary permutations and sign changes and that $\mathbb{P}[S_i = 0] = 0$ for all $1 \le i \le n$, Sparre Andersen proved in [25, Theorem C] the following *discrete arcsine law* for the position of the maximum:

$$\mathbb{P}\left[\max\{S_0, \dots, S_n\} = S_i\right] = \frac{1}{2^{2n}} \binom{2i}{i} \binom{2n-2i}{n-i}, \quad i = 0, \dots, n. \tag{1}$$

By the symmetry, the same holds for the position of the minimum. Surprisingly, the above formula is distribution-free, that is its right-hand side does not depend on the distribution of $(\xi_1, \dots, \xi_n)$ provided the symmetric exchangeability and the general position assumptions mentioned above are satisfied. Another unexpected consequence of this formula is that the maximum is more likely to be attained at $i = 0$ or $i = n$ rather than at $i \approx n/2$, as one could naïvely guess. A discussion of the arcsine laws can be found in Feller's book [6, Vol. II, Section XII.8].