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# Factorization and non-factorization theorems for pseudocontinuable functions

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ABSTRACT

Let $\theta$ be an inner function on the unit disk, and let $K^p_\theta := H^p \cap \theta\overline{H^p_0}$ be the associated star-invariant subspace of the Hardy space $H^p$, with $p \geq 1$. While a nontrivial function $f \in K^p_\theta$ is never divisible by $\theta$, it may have a factor $h$ which is "not too different" from $\theta$ in the sense that the ratio $h/\theta$ (or just the anti-analytic part thereof) is smooth on the circle. In this case, $f$ is shown to have additional integrability and/or smoothness properties, much in the spirit of the Hardy–Littlewood–Sobolev embedding theorem. The appropriate norm estimates are established, and their sharpness is discussed.



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## 1. Introduction and results

The pseudocontinuable functions in the paper's title are the noncyclic vectors of the backward shift operator

$$S^* : f \mapsto \frac{f - f(0)}{z}$$

acting on the Hardy space $H^2$ – or, more generally, $H^p$ (see the definition below) – of the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. As usual, "noncyclic" means "lying in some proper (closed) invariant subspace", and a well-known result from [7] tells us that a function $f$ is noncyclic for $S^*$ if and only if it admits a *pseudocontinuation* to $\mathbb{D}^- := \{z : |z| > 1\}$. The latter means that there exists a meromorphic function of bounded characteristic in $\mathbb{D}^-$ whose boundary values agree with $f$ almost everywhere on the circle $\mathbb{T} := \partial\mathbb{D}$. The other key notion in this paper is *smoothness*, a phenomenon that can also be described in terms of a "pseudocontinuation", this time understood as *pseudoanalytic extension* in the sense of Dyn'kin (see [16]). The general principle, as explained in [16], is that a function $f$ holomorphic on $\mathbb{D}$ (and, say, continuous up to the boundary) will be smooth on $\mathbb{T}$, in some sense or other, if and only if it extends to $\mathbb{D}^-$ as a $C^1$ function whose Cauchy–Riemann $\overline{\partial}$-derivative becomes appropriately small – or does not grow too fast – near $\mathbb{T}$.

Thus, loosely speaking, we are concerned with the interplay of the two kinds of pseudocontinuation. In this connection, we also mention the survey paper [15] which summarizes some of the previous results pertaining to the two topics and discusses the interrelationship between them.

Now let us try and describe the setup more accurately. Recall, first of all, that the Hardy space $H^p = H^p(\mathbb{D})$ with $0 < p < \infty$ is formed by those holomorphic functions $f$ on $\mathbb{D}$ which satisfy

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta) < \infty$$

(we write $m$ for the normalized arclength measure on $\mathbb{T}$), while $H^\infty = H^\infty(\mathbb{D})$ stands for the space of bounded holomorphic functions. As is customary, we identify $H^p$ functions with their boundary values (defined almost everywhere on the circle) and treat $H^p$ as a subspace of $L^p = L^p(\mathbb{T}, m)$, endowed with the standard $L^p$-norm $\|\cdot\|_p$. Further, a function $\theta \in H^\infty$ is said to be *inner* if $|\theta| = 1$ almost everywhere on $\mathbb{T}$.

Beurling's famous theorem characterizes the invariant subspaces of the (forward) shift operator $S : f \mapsto zf$ in $H^2$ as those of the form $\theta H^2$, where $\theta$ is an inner function; see, e.g., [17, Chapter II]. Accordingly, the $S^*$-invariant (or *star-invariant*) subspaces of $H^2$ can be written as

$$H^2 \ominus \theta H^2 =: K^2_\theta,$$