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Moment map flows and the Hecke correspondence for quivers ☆

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A B S T R A C T

In this paper we investigate the convergence properties of the upwards gradient flow of the norm-square of a moment map on the space of representations of a quiver. The first main result gives a necessary and sufficient algebraic criterion for a complex group orbit to intersect the unstable set of a given critical point. Therefore we can classify all of the isomorphism classes which contain an initial condition that flows up to a given critical point. As an application, we then show that Nakajima's Hecke correspondence for quivers has a Morse-theoretic interpretation as pairs of critical points connected by flow lines for the norm-square of a moment map. The results are valid in the general setting of finite quivers with relations.

1. Introduction

There is a well-known correspondence between quotients in symplectic and algebraic geometry. For example, the Kempf–Ness theorem relates GIT quotients and symplectic

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quotients of affine spaces [16] and the Donaldson–Uhlenbeck–Yau theorem relates moduli spaces of polystable holomorphic bundles to moduli spaces of Yang–Mills minima [7,8, 33].

In many examples of interest, there is also a symplectic-algebraic correspondence between GIT unstable points and critical points of a moment map functional, given by taking the limit of the downwards gradient flow of the norm-square of a moment map. This originated in Kirwan's work [18] for projective varieties, and the case of the Yang–Mills flow is studied in [5], [6] and [30]. A version of this theorem for quiver varieties is proved in [9, Thm. 3], which has since been generalised in [14] to the case of reductive group actions on affine spaces.

The goal of this paper is to further extend this correspondence between symplectic and algebraic geometry to spaces of flow lines between critical sets on the space of representations of a quiver with relations. The symplectic side of the picture determines the critical points and the flow lines for the norm-square of the moment map and one of the main theorems of this paper gives an algebraic criterion for the existence of a flow line connecting two critical points. Using this criterion we then show that critical points connected by flow lines can be interpreted in terms of the Hecke correspondence for quivers defined in [22], [23].

The setup is explained in detail in Section 2.1; here we provide a summary of the points relevant to the rest of the introduction. The vector space $\mathrm{Rep}(Q, \mathbf{v})$ of complex representations of a quiver with a fixed dimension vector $\mathbf{v}$ has a natural symplectic structure determined by the Hermitian structure on the complex vector space at each vertex of the quiver. Associated to this is a Hamiltonian action of a group $K_{\mathbf{v}}$ and a moment map which we denote by μ. The complexification of $K_{\mathbf{v}}$ is denoted $G_{\mathbf{v}}$. In the paper [9] we studied the downwards gradient flow of $\|\mu - \alpha\|^2$ for any stability parameter α and showed the existence of a Morse stratification which coincides with the algebraic Harder–Narasimhan stratification defined by Geometric Invariant Theory (cf. [28]). The main theorem of [9] identifies the isomorphism class of the limit of the downwards flow with the graded object of the Harder–Narasimhan–Jordan–Hölder filtration associated to the initial condition.

The entire setup described above restricts to any closed subset $Z \subset \mathrm{Rep}(Q, \mathbf{v})$ which is preserved by the action of $G_{\mathbf{v}}$. An important special case of this is the variety of representations of a quiver satisfying a finite set of relations in the path algebra. Moreover, the critical sets of the norm-square of the moment map have a simple interpretation in terms of representations which are the direct sum of subrepresentations of different slopes minimising the norm-square of the moment map (cf. (2.15)). Therefore the critical sets can be classified by the dimension and slope of each of the subrepresentations in this splitting in analogy with the classification of the critical sets of the Yang–Mills functional described by Atiyah and Bott in [1].

Given a critical point x, let W_x^- denote the *unstable set* of initial conditions for which the upwards flow of $\|\mu - \alpha\|^2$ converges to x. Associated to the critical point is another space called the *negative slice* S_x^-, which is defined in terms of a group action around