Advances in Mathematics 320 (2017) 827–875

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Sharp bounds for the cubic Parsell–Vinogradov system in two dimensions ☆

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A R T I C L E   I N F O

Article history:
Received 29 August 2016
Received in revised form 12 August 2017
Accepted 5 September 2017
Communicated by Charles Fefferman

MSC:
primary 11L07
secondary 42A45

Keywords:
Parsell–Vinogradov systems
Brascamp–Lieb inequalities
Transversality conditions

A B S T R A C T

We prove a sharp decoupling for a certain two dimensional surface in $\mathbb{R}^9$. As an application, we obtain the full range of expected estimates for the cubic Parsell–Vinogradov system in two dimensions.

☆ The first author is partially supported by the NSF grant DMS-1301619. The second author is partially supported by the NSF Grant DMS-1161752.

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http://dx.doi.org/10.1016/j.aim.2017.09.008

1. Introduction and statements of main results

Fix $d, s \geq 1$ and $k \geq 2$. We use $\mathbf{x}$ to denote the vector $(x_1, \dots, x_d) \in \mathbb{R}^d$, and $\mathbf{i}$ to denote the d-tuple $(i_1, \dots, i_d)$ of non-negative integers. The monomial $x_1^{i_1} \dots x_d^{i_d}$ will be abbreviated to $\mathbf{x}^{\mathbf{i}}$. Consider the integer solutions

$$\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s, \mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_s \tag{1.1}$$

of the system of Diophantine equations (often referred to as the *Parsell–Vinogradov* system)

$$\mathbf{x}_1^{\mathbf{i}} + \cdots + \mathbf{x}_s^{\mathbf{i}} = \mathbf{y}_1^{\mathbf{i}} + \cdots + \mathbf{y}_s^{\mathbf{i}}. \tag{1.2}$$

Here $0 \leq i_1, i_2, ..., i_d \leq k$ range through all possible integers such that $1 \leq i_1 + i_2 + ... + i_d \leq k$. For instance, when $d = 1$, the system (1.2) consists of the following k equations, known as the classical Vinogradov system

$$x_1^i + \cdots + x_s^i = y_1^i + \cdots + y_s^i, \text{ with } 1 \leq i \leq k. \tag{1.3}$$

When $d = k = 2$, the system (1.2) becomes

$$\begin{aligned} x_{1,1} + x_{1,2} + ... + x_{1,s} &= y_{1,1} + y_{1,2} + ... + y_{1,s}, \\ x_{2,1} + x_{2,2} + ... + x_{2,s} &= y_{2,1} + y_{2,2} + ... + y_{2,s}, \\ x_{1,1}^2 + x_{1,2}^2 + ... + x_{1,s}^2 &= y_{1,1}^2 + y_{1,2}^2 + ... + y_{1,s}^2, \\ x_{2,1}^2 + x_{2,2}^2 + ... + x_{2,s}^2 &= y_{2,1}^2 + y_{2,2}^2 + ... + y_{2,s}^2, \\ x_{1,1}x_{2,1} + x_{1,2}x_{2,2} + ... + x_{1,s}x_{2,s} &= y_{1,1}y_{2,1} + y_{1,2}y_{2,2} + ... + y_{1,s}y_{2,s}. \end{aligned} \tag{1.4}$$

For a large N, we let $J_{s,d,k}(N)$ denote the number of integer solutions (1.1) of the system of equations (1.2) satisfying $1 \leq x_{1,j}, ..., x_{d,j}, y_{1,j}, ..., y_{d,j} \leq N$ for each $1 \leq j \leq s$.

As far as we can tell, the investigation of the quantities $J_{s,d,k}(N)$ for $d \geq 2$ was initiated by Parsell in [12]. This paper also explains some of the motivation behind considering such quantities. For instance, one motivation comes from counting rational linear subspaces of a given dimension lying on the hyper-surface defined by

$$c_1 z_1^k + \cdots + c_s z_s^k = 0, \tag{1.5}$$

for given $c_1, \dots, c_s \in \mathbb{Z}$. In order to apply the Hardy–Littlewood circle method, one needs a good upper bound for $J_{s,d,k}(N)$. We mention that a related Diophantine system was considered earlier in [1].

Parsell, Prendiville and Wooley [13] provided the following lower bound for $J_{s,d,k}(N)$.