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On polynomially integrable convex bodies

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An infinitely smooth convex body in $\mathbb{R}^n$ is called polynomially integrable of degree N if its parallel section functions are polynomials of degree N. We prove that the only smooth convex bodies with this property in odd dimensions are ellipsoids, if $N \geq n-1$. This is in contrast with the case of even dimensions and the case of odd dimensions with $N < n-1$, where such bodies do not exist, as it was recently shown by Agranovsky.

1. Introduction

Let K be an infinitely smooth convex body in $\mathbb{R}^n$. The parallel section function of K in the direction $\xi \in S^{n-1}$ is defined by

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$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap \{(x,\xi) = t\}) = \int\limits_{(x,\xi)=t} \chi_K(x)dx, \quad t \in \mathbb{R},$$

where χ_K is the indicator function of K, and (x,ξ) is the scalar product in $\mathbb{R}^n$.

It is clear that if B is the Euclidean ball of radius r centered at the origin, then

$$A_{B,\xi}(t) = c_n(r^2 - t^2)^{(n-1)/2},$$

for $|t| \le r$, where c_n is a constant depending on n only. In particular, if n is odd then the parallel section function of B is a polynomial of t for every $\xi \in S^{n-1}$ and every t for which $K \cap \{x : (x,\xi) = t\}$ is non-empty. This property also holds for ellipsoids.

Definition 1.1. A convex body K (or more generally, a bounded domain) in $\mathbb{R}^n$ is called *polynomially integrable* (of degree N) if

$$A_{K,\xi}(t) = \sum_{k=0}^{N} a_k(\xi)\ t^k \tag{1.1}$$

for some integer N, all $\xi \in S^{n-1}$ and all t for which the set $K \cap \{x : (x,\xi) = t\}$ is non-empty. Here, a_k are functions on the sphere. We assume that the function a_N is not identically zero.

This concept was introduced by Agranovsky in [1]. He also established a number of properties of such bodies. In particular, he showed that there are no bounded polynomially integrable domains with smooth boundaries in Euclidean spaces of even dimensions. In odd dimensions he proved that polynomially integrable bounded domains with smooth boundaries are convex, and that there are no polynomially integrable bounded domains in $\mathbb{R}^n$ with smooth boundaries of degree strictly less than $n-1$, while every such body with degree $n-1$ is an ellipsoid. For polynomially integrable domains of higher degrees Agranovsky asks the following.

Problem 1.2. Is it true that in the odd-dimensional space the only polynomially integrable domains are ellipsoids?

Problems of this kind go back to Newton [11]. Consider the volume of the "halves" of the body cut off by the hyperplane $(x,\xi) = t$, that is $V^+_{K,\xi}(t) = \int_t^\infty A_{K,\xi}(z)dz$ and $V^-_{K,\xi}(t) = \int_{-\infty}^t A_{K,\xi}(z)dz$. A body K is called algebraically integrable if there is a polynomial F such that $F(\xi_1, \dots, \xi_n, t, V^\pm_{K,\xi}(t)) = 0$ for every choice of parameters ξ and t. Newton showed that in $\mathbb{R}^2$ there are no algebraically integrable convex bodies with infinitely smooth boundaries. Arnold asked for extensions of Newton's result to other dimensions and general domains; see problems 1987-14, 1988-13, and 1990-27 in [2]. Vassiliev [14] generalized Newton's result by showing that there are no algebraically integrable bounded domains with infinitely smooth boundary in $\mathbb{R}^n$ for even n.