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Multilinear Schur multipliers and Schatten properties of operator Taylor remainders

Denis Potapov [a,*,1], Anna Skripka [b,2], Fedor Sukochev [a,1], Anna Tomskova [a,1]

[a] *School of Mathematics and Statistics, University of New South Wales, Kensington, NSW 2052, Australia*
[b] *Department of Mathematics and Statistics, University of New Mexico, 400 Yale Blvd NE, MSC01 1115, Albuquerque, NM 87131, USA*

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ABSTRACT

We establish sharp conditions on scalar functions and perturbations that guarantee Schatten summability of nth order operator Taylor remainders. In the special case of dimension one, our estimates of these remainders deliver well known classical estimates of scalar Taylor remainders. We prove that if a scalar function f is in the set C^n and a perturbation is in the pth Schatten class $\mathcal{S}^p$, $p > n$, then the respective nth order operator Taylor remainder is an element of $\mathcal{S}^{p/n}$ and has an estimate like the one in [16]. We construct examples of $f \in C^n$ and perturbations in $\mathcal{S}^n$ such that the nth order Taylor remainder of the respective operator function is not in $\mathcal{S}^1$. Our construction relies, in particular, on novel dimension dependent estimates for Schatten norms of multilinear Schur multipliers from below that are of interest in their own right. Our results apply to both self-adjoint and unitary operators.

* Corresponding author.
E-mail addresses: d.potapov@unsw.edu.au (D. Potapov), skripka@math.unm.edu (A. Skripka), f.sukochev@unsw.edu.au (F. Sukochev), a.tomskova@unsw.edu.au (A. Tomskova).
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1. Introduction

Let A and B be bounded self-adjoint operators and let U be a unitary operator on an infinite dimensional Hilbert space $\mathcal{H}$. Let f and φ be C^n-functions on the real line $\mathbb{R}$ and on the unit circle $\mathbb{T}$, respectively. This paper presents sharp conditions on f, φ, and B such that the nth Taylor remainder

$$R_{n,f,A}(B) := f(A+B) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k}{dt^k}\bigg|_{t=0} f(A+tB) \tag{1.1}$$

in the self-adjoint case or

$$Q_{n,\varphi,U}(B) := \varphi(e^{iB}U) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k}{dt^k}\bigg|_{t=0} \varphi(e^{itB}U) \tag{1.2}$$

in the unitary case belongs to a Schatten class. It also presents sharp estimates that these remainders satisfy.

Our first main result (Theorem 4.1) shows that when the perturbation B is in the Schatten class $\mathcal{S}^p$ with $p > n$ and f is in $C^n(\mathbb{R})$, then the remainder $R_{n,f,A}(B)$ belongs to $\mathcal{S}^{p/n}$ and has an estimate analogous to the classical estimate for functions of scalars.

Our second main result (Theorem 5.1) shows that the above assumption on B to be in $\mathcal{S}^p$ with $p > n$ is sharp, that is, there exist a C^n-function f and self-adjoint bounded operators A and $B \in \mathcal{S}^n$ such that $R_{n,f,A}(B) \notin \mathcal{S}^1$.

Completely analogous results are also established for $Q_{n,\varphi,U}(B)$ in Theorems 4.2 and 5.11.

The results of Theorems 4.1 and 4.2 substantially sharpen up to date conditions on f and φ guaranteeing

$$R_{n,f,A},\ Q_{n,\varphi,U} : \mathcal{S}^p \mapsto \mathcal{S}^{p/n}, \quad p > n. \tag{1.3}$$

It followed from [14] and [18] that (1.3) holds for f in the intersection of Besov classes $B^n_{\infty 1}(\mathbb{R}) \cap B^1_{\infty 1}(\mathbb{R})$ and φ in $B^n_{\infty 1}(\mathbb{T})$, while we prove (1.3) for $f \in C^n(\mathbb{R})$ and $\varphi \in C^n(\mathbb{T})$. The estimates for the remainders obtained in Theorems 4.1 and 4.2 extend the respective estimates in [3,1,12,16,17,15,19] to arbitrary $f \in C^n(\mathbb{R})$ and $\varphi \in C^n(\mathbb{T})$, $n \in \mathbb{N}$.

The results of Theorems 5.1 and 5.11 substantially extend the results of [5,6], where only the special case $n = 2$ was treated. Techniques of this paper ensure a unified treatment of all self-adjoint and unitary cases for every $n \geqslant 2$. Observe also that the counterexample in the case $n = 1$ is constructed in [11] (see also [8,13,24]). We note, in passing, that the question that motivated the example in [9] also led to the introduction of so-called operator-Lipschitz functions, whose detailed study can be found in the recent survey [2].

In Theorem 2.3 we establish an estimate from below for a Schatten norm of a multilinear Schur multiplier via an estimate for a linear one. This reduction implies, in