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Random walks and induced Dirichlet forms on self-similar sets ☆

Shi-Lei Kong [a,*], Ka-Sing Lau [a], Ting-Kam Leonard Wong [b]

[a] *Department of Mathematics, The Chinese University of Hong Kong, Hong Kong*
[b] *Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA*

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ABSTRACT

Let K be a self-similar set satisfying the open set condition. Following Kaimanovich's elegant idea [25], it has been proved that on the symbolic space X of K a natural augmented tree structure $\mathfrak{E}$ exists; it is hyperbolic, and the hyperbolic boundary $\partial_H X$ with the Gromov metric is Hölder equivalent to K. In this paper we consider certain reversible random walks with return ratio $0 < \lambda < 1$ on $(X, \mathfrak{E})$. We show that the Martin boundary $\mathcal{M}$ can be identified with $\partial_H X$ and K. With this setup and a device of Silverstein [41], we obtain precise estimates of the Martin kernel and the Naïm kernel in terms of the Gromov product. Moreover, the Naïm kernel turns out to be a jump kernel satisfying the estimate $\Theta(\xi, \eta) \asymp |\xi - \eta|^{-(\alpha+\beta)}$, where α is the Hausdorff dimension of K and β depends on λ. For suitable β, the kernel defines a regular non-local Dirichlet form on K. This extends the results of Kigami [27] concerning random walks on certain trees with Cantor-type sets as boundaries (see also [5]).

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* Corresponding author.
E-mail addresses: slkong@math.cuhk.edu.hk (S.-L. Kong), kslau@math.cuhk.edu.hk (K.-S. Lau), tkleonardwong@gmail.com (T.-K.L. Wong).

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Contents

1. Introduction

Let $\mathbb{D}$ be the open unit disk, and let $\mathbb{T}$ be the boundary circle parameterized by $\{\theta : 0 \leq \theta < 2\pi\}$. Let

$$\mathcal{E}_{\mathbb{D}}(u,v) = \int_{\mathbb{D}} \nabla u(x) \nabla v(x) dx \tag{1.1}$$

be the standard Dirichlet form on $\mathbb{D}$. In classical analysis, it is well-known that a function $\varphi \in L^1(\mathbb{T})$ can be extended to a harmonic functions on $\mathbb{D}$ via the Poisson integral

$$(H\varphi)(x) = \int_{\mathbb{T}} \varphi(\theta) K(x,\theta) d\theta, \qquad x \in \mathbb{D},$$

where $K(x,\theta)$ is the Poisson kernel. Furthermore, there is an induced Dirichlet form on $\mathbb{T}$ defined by

$$\mathcal{E}_{\mathbb{T}}(\varphi,\psi) = \mathcal{E}_{\mathbb{D}}(H\varphi, H\psi).$$

Indeed, it can be shown that

$$\mathcal{E}_{\mathbb{T}}(\varphi,\psi) = \frac{1}{16\pi} \int_{\mathbb{T}} \int_{\mathbb{T}} (\varphi(\theta) - \varphi(\theta'))(\psi(\theta) - \psi(\theta')) \frac{1}{\sin^2(\frac{\theta-\theta'}{2})} \, d\theta d\theta'. \tag{1.2}$$

This integral is called the *Douglas integral* (see [14, Section 1.2]). From the probabilistic point of view, the Dirichlet form in (1.1) is associated with a Brownian motion on $\mathbb{D}$. The hitting distribution of the Brownian motion at the boundary $\mathbb{T}$ (starting from 0) is the uniform distribution $\frac{d\theta}{2\pi}$; the induced Dirichlet form in (1.2) corresponds to the reflecting Brownian motion on $\overline{\mathbb{D}}$ time-changed by its local time on $\mathbb{T}$, and defines a jump process on $\mathbb{T}$ which is a Cauchy process [6].

The above consideration has a counterpart in Markov chain theory. Let $\{Z_n\}_{n=0}^{\infty}$ be a transient Markov chain on an infinite discrete set X with transition probability P.