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Asymptotic ∗-moments of some random Vandermonde matrices

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ABSTRACT

Appropriately normalized square random Vandermonde matrices based on independent random variables with uniform distribution on the unit circle are studied. It is shown that as the matrix sizes increases without bound, with respect to the expectation of the trace there is an asymptotic ∗-distribution, equal to that of a $C[0,1]$-valued R-diagonal element.

1. Introduction

We consider the random Vandermonde matrix X_N, whose (i,j)-th entry is $N^{-1/2}\zeta_i^j$, where $\zeta_1,\ldots,\zeta_N$ are independent with Haar measure distribution on the unit circle.

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These have been studied in [6], [7], [10] and [11] and are of interest for applications in finance, signal array processing, wireless communications and biology (see [6] for references). In [6], Ryan and Debbah show that asymptotic moments of $X_N^*X_N$ (namely, the limits

$$\lim_{N\to\infty} \mathbb{E}\circ \mathrm{tr}((X_N^*X_N)^p),$$

where $\mathbb{E}$ is the expectation and tr is the normalized trace on matrix algebras), exist and are given by sums of volumes of certain polytopes. They also compute some of these asymptotic moments. In [10], Tucci and Whiting show among other things that the asymptotic moments are given by

$$\lim_{N\to\infty} \mathbb{E}\circ \mathrm{tr}((X_N^*X_N)^p) = \int x^p \, d\mu(x)$$

for a unique measure μ on $[0,\infty)$ with unbounded support. (This uses the Stieltjes solution to the moment problem and a theorem of Carleman — for the former, see p. 76 of [1].) Further results are proved in [7] and [11].

G. Tucci asked [9] whether X_N is asymptotically R-diagonal with respect to the expectation of the trace. In this paper, we answer Tucci's question negatively, but show that X_N has an asymptotic $*$-distribution as $N\to\infty$, which is in fact the $*$-distribution of an element that is R-diagonal over the C*-algebra $C[0,1]$.

To be precise, we show that, for all $n\in\mathbf{N}$ and all $\epsilon(1),\ldots,\epsilon(n)\in\{1,*\}$,

$$\lim_{N\to\infty} \mathbb{E}\circ \mathrm{tr}(X_N^{\epsilon(1)}\cdots X_N^{\epsilon(n)})$$

exists and we describe this limiting $*$-moment using the notion of $C[0,1]$-valued R-diagonality.

Usual (or scalar-valued) R-diagonal elements are very natural in free probability theory, and have been much studied; they were introduced by Nica and Speicher in [5]. The algebra-valued version was introduced by Śniady and Speicher in [8] and has been further studied in [3]. We will give the definition from [3], which is an easy reformulation of one of the characterizations in [8].

The setting for algebra-valued R-diagonal elements is a B-valued $*$-noncommutative probability space $(A,\mathcal{E})$, where $B\subseteq A$ is a unital inclusion of unital $*$-algebras and $\mathcal{E}:A\to B$ is a conditional expectation, namely, a B-bimodular unital projection.

Definition 1.1. Given $n\in\mathbf{N}$ and $\epsilon=(\epsilon(1),\ldots,\epsilon(n))\in\{1,*\}^n$, we define the *maximal alternating interval partition* $\sigma(\epsilon)$ to be the interval partition of $\{1,\ldots,n\}$ whose blocks are the maximal interval subsets I of $\{1,\ldots,n\}$ such that if $j\in I$ and $j+1\in I$, then $\epsilon(j)\neq\epsilon(j+1)$.

For example, if $\epsilon=\{1,1,*,1,*,*\}$, then $\sigma(\epsilon)=\{\{1\},\{2,3,4,5\},\{6\}\}$.