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Convex body domination and weighted estimates with matrix weights

Fedor Nazarov [a], Stefanie Petermichl [b], Sergei Treil [c,1], Alexander Volberg [d,*,2]

[a] *Department of Mathematics, Kent State University, USA*
[b] *Department of Mathematics, Université Paul Sabatier, Toulouse, France*
[c] *Department of Mathematics, Brown University, USA*
[d] *Department of Mathematics, Michigan Sate University, East Lansing, MI 48823, USA*

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ABSTRACT

We introduce the so called *convex body valued* sparse operators, which generalize the notion of sparse operators to the case of spaces of vector valued functions.
We prove that Calderón–Zygmund operators as well as Haar shifts and paraproducts can be dominated by such operators. By estimating sparse operators we obtain weighted estimates with matrix weights. We get two weight A_2–A_∞ estimates, that in the one weight case give us the estimate

$$\|T\|_{L^2(W)\to L^2(W)} \le C[W]_{\mathbf{A}_2}^{1/2}[W]_{A_\infty} \le C[W]_{\mathbf{A}_2}^{3/2}$$

where T is either Calderón–Zygmund operator (with modulus of continuity satisfying the Dini condition), or a Haar shift or a paraproduct.

* Corresponding author.
E-mail addresses: nazarov@math.kent.edu (F. Nazarov), stefanie.petermichl@gmail.com (S. Petermichl), treil@math.brown.edu (S. Treil), volberg@math.msu.edu (A. Volberg).

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Notation

$|Q|$ for $Q \subset \mathbb{R}^N$ denotes its N-dimensional Lebesgue measure;

$\mathscr{D}$ a dyadic lattice. We consider all "translations" of the standard dyadic lattice;

$\langle f \rangle_Q$ average of the function f over Q, $\langle f \rangle_Q := |Q|^{-1} \int_Q f(x)\mathrm{d}x$;

$\langle\!\langle f \rangle\!\rangle_Q$ "convex body valued" average of a functions f with values in $\mathbb{R}^d$, see Section 2.2;

$\| \cdot \|, \| \cdot \|$ norm; since we are dealing with matrix- and operator-valued functions we will use the symbol $\| \cdot \|$ (usually with a subscript) for the norm in a functions space, while $\| \cdot \|$ is used for the norm in the underlying vector (operator) space. Thus for a vector-valued function f the symbol $\|f\|_2$ denotes its L^2-norm, but the symbol $\|f\|$ stands for the scalar-valued function $x \mapsto \|f(x)\|$;

1. Motivations, definitions and results

This paper started as an (unsuccessful) attempt to prove the so-called A_2-conjecture for the weighted estimates with matrix weights.

Recall that a (d-dimensional) matrix weight on $\mathbb{R}^N$ is a locally integrable function on $\mathbb{R}^N$ with values in the set of positive-semidefinite $d \times d$ matrices. The weighted space $L^2(W)$ is defined as the space of all measurable functions $f : \mathbb{R}^N \to \mathbb{F}^d$, (here $\mathbb{F} = \mathbb{R}$, or $\mathbb{F} = \mathbb{C}$) for which

$$\|f\|^2_{L^2(W)} := \int (W(x)f(x), f(x))\mathrm{d}x < \infty\,;$$

here $(\cdot, \cdot)$ means the usual duality in $\mathbb{F}^d$.

A matrix weight W is said to satisfy the matrix $\mathbf{A}_2$ condition (write $W \in (\mathbf{A}_2)$) if

$$[W]_{\mathbf{A_2}} := \sup_Q \| \langle W \rangle_Q^{1/2} \langle W^{-1} \rangle_Q^{1/2} \|^2 < \infty\,.$$

The quantity $[W]_{\mathbf{A_2}}$ is called the $\mathbf{A}_2$ *characteristic of the weight* W.