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Ancient solutions of the Ricci flow on bundles

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We generalize the circle bundle examples of ancient solutions of the Ricci flow discovered by Bakas, Kong, and Ni to a class of principal torus bundles over an arbitrary finite product of Fano Kähler–Einstein manifolds studied by Wang and Ziller in the context of Einstein geometry. As a result, continuous families of κ-collapsed and κ-noncollapsed ancient solutions of type I are obtained on circle bundles for all odd dimensions ≥ 7. In dimension 7 such examples moreover exist on pairs of homeomorphic but not diffeomorphic manifolds. Continuous families of κ-collapsed ancient solutions of type I are also obtained on torus bundles for all dimensions ≥ 8.

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1. Introduction

An ancient solution $g(t)$ of the Ricci flow in this paper is one that exists on a half infinite interval of the type $(-\infty, T)$ for some finite time T and with $g(t)$ complete for each t. These solutions are important because they arise naturally when finite time singular solutions of the flow are blown up. Rigidity phenomena for ancient solutions have been established by L. Ni ([25]), and by S. Brendle, G. Huisken, and C. Sinestrari ([6]) under certain non-negativity conditions on the full curvature tensor. For dimension two, classification theorems were achieved by P. Daskalopoulos, R. Hamilton, and N. Sesum ([13]) for closed surfaces and by R. Hamilton ([19]), S.C. Chu ([12]), and P. Daskalopoulos and N. Sesum ([14]) for complete non-compact surfaces.

Examples of ancient solutions in higher dimensions do exist. The first examples (in dimension 3) were due to Fateev [16] and Perelman [28]. Other examples in dimension 3 result from the work of X.D. Cao, J. Guckenheimer, and L. Saloff-Coste ([11], [10]) on the backwards behavior of the Ricci flow on locally homogeneous 3-manifolds. In [1] I. Bakas, S. L. Kong, and L. Ni took the limit of a special family of the Fateev examples and generalized it to give examples of ancient solutions on odd-dimensional spheres, complex projective spaces, and the twistor spaces of compact quaternion-Kähler manifolds. By studying a system of two ODEs J. Lauret ([24]) gave explicit examples of 2-parameter families of left-invariant ancient solutions on any compact simple Lie group except $Sp(2k+1)$. Further homogeneous examples can be found in [8] and [5]. Finally, we mention two inhomogeneous examples in dimension 4: a compact ancient solution due to S. Brendle and N. Kapouleas ([7]) which is constructed by desingularizing an orbifold quotient of the flat torus, and a non-compact inhomogeneous example due to R. Takahashi ([29]) which has the Euclidean Schwarzschild metric as backwards limit.

In this article we shall be concerned with the construction of ancient solutions on closed manifolds which are not highly symmetric except in special cases. Specifically, we shall generalize the circle bundle examples in [1, Theorem 5.1] to a class of principal torus bundles over an arbitrary finite product of Fano KE (short for Kähler–Einstein) manifolds. One may view our work here as a dynamic version of the work in [32], although the aim, of course, is not to reconstruct the Einstein metrics obtained in that work via the Ricci flow.

To describe our main results, let P_Q denote a principal r-torus bundle with Euler classes Q over a product of m arbitrary KE manifolds (M_i, g_i) with positive first Chern class. As in [32] the integral cohomology classes in Q are taken to be allowable rational linear combinations of the first Chern classes of M_i subject to certain non-degeneracy conditions explained in detail in §2. We consider the class of connection-type metrics on P_Q determined by an arbitrary left-invariant metric on the torus fibers, arbitrary scalings of the KE metrics g_i by constants, and the harmonic representatives of the Euler classes in Q with respect to the base metric chosen. This class of metrics is preserved by the Ricci flow, and the Ricci flow equation simplifies to a system of nonlinear ODEs, which we analyze in some detail to deduce