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On the cohomology of Fano varieties and the Springer correspondence

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with an appendix by Dennis Stanton

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ABSTRACT

In this paper we compute the cohomology of the Fano varieties of k-planes in the smooth complete intersection of two quadrics in $\mathbb{P}^{2g+1}$, using Springer theory for symmetric spaces.

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1. Introduction

In this paper we compute the cohomology of the Fano varieties Fano_k of k-planes in the smooth complete intersection of two quadrics in $\mathbb{P}^{2g+1}$, with $g \geq 1$. These Fano varieties have a concrete interpretation as moduli spaces of vector bundles (with extra structure) on a hyperelliptic curve C of genus g. When $k = g-1$ then $\mathrm{Fano}_{g-1} = \mathrm{Jac}(C)$, the Jacobian of C [18,9] and when $k = g-2$ then $\mathrm{Fano}_{g-2} = \mathrm{Bun}_2(C)$, the moduli space of stable rank 2 vector bundles on C with fixed odd determinant [8]. For $k < g-2$ a more elaborated interpretation of the varieties Fano_k as moduli spaces of bundles is given in [16]. The curve C arises from the intersection of two quadrics in the following manner. If the intersection of the two quadrics is given by the pencil $\mu Q_1 + \lambda Q_2$ then C is the hyperelliptic curve over $\mathbb{P}^1$ ramified at the points $[\mu, \lambda]$ where the quadric $\mu Q_1 + \lambda Q_2$ becomes singular.

The goal of this paper is to describe the cohomology of the varieties Fano_k in general. The form of our answer is in the spirit of the main theorem of [14] who, from our point of view, treats the case Fano_{g-2}. He makes use of the mapping class group which for us, as we work with hyperelliptic curves, is replaced by the fundamental group of the universal family of hyperelliptic curves.

To state our result note that $\dim \mathrm{Fano}_{g-i} = (g-i+1)(2i-1)$. We also write

$$\bar{H}^k(\mathrm{Fano}_{g-i}, \mathbb{C}) = H^{\dim \mathrm{Fano}_{g-i} - k}(\mathrm{Fano}_{g-i}, \mathbb{C}), \quad \bar{\wedge}^k\left(H^1(C, \mathbb{C})\right) = \wedge^{g-k}\left(H^1(C, \mathbb{C})\right).$$

Theorem 1.1. *For $i \geq 2$, we have*

$$\bar{H}^k(\mathrm{Fano}_{g-i}, \mathbb{C}) \cong \bigoplus_{j=i-1}^{g} N_i(k,j)\, \bar{\wedge}^j\left(H^1(C, \mathbb{C})\right),$$

where $N_i(k,j)$ is the coefficient of q^k in

$$q^{-(j-i+1)(2i-1)}(1-q^{4j})\frac{\prod_{l=j-i+2}^{i+j-2}(1-q^{2l})}{\prod_{l=1}^{2i-2}(1-q^{2l})}.$$

In particular, the numbers $N_i(k,j)$ are independent of the genus g.

To come up with this formula we were inspired by the case of $\mathrm{Fano}_{g-2} = \mathrm{Bun}_2(C)$ treated by Nelson [14]. He, in turn, states that the formula in the case of $\mathrm{Fano}_{g-2} = \mathrm{Bun}_2(C)$ was conjectured by Donaldson. We guessed the general formula above after reading [23] augmented by some experimentation. By our methods we reduced the proof of the theorem to a combinatorial identity which expresses the Poincare polynomial of the orthogonal Grassmannian in terms of the Poincare polynomials of ordinary Grassmannians in a particular manner, see formula (3.13). This combinatorial identity was proved by Stanton. His proof is included in this paper as an appendix. We have not been