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Homological stability for automorphism groups

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ABSTRACT

Given a family of groups admitting a braided monoidal structure (satisfying mild assumptions) we construct a family of spaces on which the groups act and whose connectivity yields, via a classical argument of Quillen, homological stability for the family of groups. We show that stability also holds with both polynomial and abelian twisted coefficients, with no further assumptions. This new construction of a family of spaces from a family of groups recovers known spaces in the classical examples of stable families of groups, such as the symmetric groups, general linear groups and mapping class groups. By making systematic the proofs of classical stability results, we show that they all hold with the same type of coefficient systems, obtaining in particular without any further work new stability theorems with twisted coefficients for the symmetric groups, braid groups, automorphisms of free groups, unitary groups, mapping class groups of non-orientable surfaces and mapping class groups of 3-manifolds. Our construction can also be applied to families of groups not considered before in the context of homological stability.

As a byproduct of our work, we construct the braided analogue of the category FI of finite sets and injections relevant to the present context, and define polynomiality for functors in the context of pre-braided monoidal categories.

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Contents

A family of groups

$$G_1 \hookrightarrow G_2 \hookrightarrow \cdots \hookrightarrow G_n \hookrightarrow \cdots$$

is said to satisfy *homological stability* if the induced maps

$$H_i(G_n) \longrightarrow H_i(G_{n+1})$$

are isomorphisms in a range $0 \leq i \leq f(n)$ increasing with n. In this paper, we prove that homological stability always holds if there is a monoidal category $\mathcal{C}$ satisfying a certain hypothesis and a pair of objects A and X in $\mathcal{C}$, such that G_n is the group of automorphisms of $A \oplus X^{\oplus n}$ in $\mathcal{C}$. We show that stability holds not just for constant coefficients, but also for both *polynomial* and *abelian* coefficients, without any further assumption on $\mathcal{C}$.

The polynomial coefficient systems considered here are functors $F : \mathcal{C} \to \mathbb{Z}$-Mod satisfying a finite degree condition. They are generalisations of polynomial functors