ARTICLE IN PRESS YAIMA:5738

Advances in Mathematics • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

A symplectic proof of the Horn inequalities

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A R T I C L E   I N F O

Article history:
Received 20 October 2014
Received in revised form 4 January 2017
Accepted 5 January 2017
Available online xxxx
Communicated by Tomasz S. Mrowka

Keywords:
Horn inequalities
Tropical approximation
Poisson–Lie groups
Planar networks

A B S T R A C T

In this paper, we give a symplectic proof of the Horn inequalities on eigenvalues of a sum of two Hermitian matrices with given spectra. Our method is a combination of tropical calculus for matrix eigenvalues, combinatorics of planar networks, and estimates for the Liouville volume. As a corollary, we give a tropical description of the Duistermaat–Heckman measure on the Horn polytope.

1. Introduction

1.1. The Horn problem

Fix a positive integer n, and let $\mathcal{H}$ be the set of Hermitian matrices of size n. For $K \in \mathcal{H}$, denote by $\lambda(K) = (\lambda_1 \geq \lambda_2, \geq \cdots \geq \lambda_n)$ the set of eigenvalues of K listed in decreasing order, and introduce the map $l : \mathcal{H} \to \mathbb{R}^n$ defined by the equalities

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http://dx.doi.org/10.1016/j.aim.2017.01.006

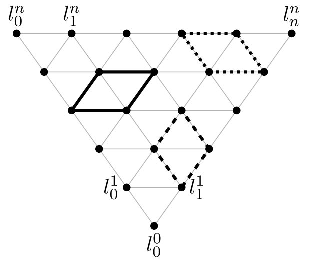

Fig. 1. The triangular tableau with three types of rhombi.

$$l_1(K) = \lambda_1,\ l_2(K) = \lambda_1 + \lambda_2,\ \dots,\ l_n(K) = \lambda_1 + \cdots + \lambda_n = \mathrm{Tr}(K).$$

We will call the set

$$\mathcal{C}_{\mathrm{Horn}} = \{(a,b,c) \in \mathbb{R}^{3n};\ \exists (K_1, K_2) \in \mathcal{H}^{\times 2} : \\ l(K_1) = a, l(K_2) = b, l(K_1 + K_2) = c\} \tag{1}$$

the *Horn cone.*

Clearly, $\mathcal{C}_{\mathrm{Horn}}$ is a closed subset of the hyperplane

$$\{(a,b,c) \in \mathbb{R}^{3n}; a_n + b_n = c_n\} \subset \mathbb{R}^{3n},$$

and $\tau\mathcal{C}_{\mathrm{Horn}} = \mathcal{C}_{\mathrm{Horn}}$ for any $\tau > 0$.

The problem of determining this cone, known as the Horn problem, has a long history (see [9] for details). The first conjectural description was given by Horn [11] in 1962; it presents $\mathcal{C}_{\mathrm{Horn}}$ as the set of solutions of a complicated, recursively defined list of linear inequalities. This description, in particular, implies that $\mathcal{C}_{\mathrm{Horn}}$ is a closed convex cone. Later, a natural explanation for this fact was found in terms of convexity properties of moment maps in symplectic geometry.

In 1999, Knutson and Tao came up with the following much simpler, albeit implicit description of $\mathcal{C}_{\mathrm{Horn}}$ ([13], see also [5,14]). Consider the regular triangulation of order n of an equilateral triangle. The triangle is divided into n^2 small triangles. Two adjacent triangles form a rhombus, which can be of one of the three types shown in Fig. 1.

We will call the assignment of a real number to each of the nodes of the triangulation a *tableau*. Denoting by ∇ the set of nodes of the triangulation, we can identify the space of tableaux with $\mathbb{R}^{\nabla}$.

Let l_i^k be the number at the ith node in the kth row of the triangulation, $0 \le i \le k \le n$. Then each rhombus gives rise to an inequality: the sum of the two numbers assigned to the endpoints of the short diagonal is greater than or equal to the sum of the two numbers assigned to the endpoints of the long diagonal. A tableau is called a *hive* if it satisfies all the inequalities, i.e., if for $0 < i \le k < n$,

Please cite this article in press as: A. Alekseev et al., A symplectic proof of the Horn inequalities, Adv. Math. (2017), http://dx.doi.org/10.1016/j.aim.2017.01.006