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Geometric representations of the formal affine Hecke algebra

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ABSTRACT

For any formal group law, there is a formal affine Hecke algebra defined by Hoffnung–Malagón-López–Savage–Zainoulline. Coming from this formal group law, there is also an oriented cohomology theory. We identify the formal affine Hecke algebra with a convolution algebra coming from the oriented cohomology theory applied to the Steinberg variety. As a consequence, this algebra acts on the corresponding cohomology of the Springer fibers. This generalizes the action of classical affine Hecke algebra on the K-theory of the Springer fibers constructed by Lusztig. We also give a residue interpretation of the formal affine Hecke algebra, which generalizes the residue construction of Ginzburg–Kapranov–Vasserot when the formal group law comes from a 1-dimensional algebraic group.

Contents

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0. Introduction

Affine Hecke-type algebras arise from the study of representations of Chevalley groups over $\mathfrak{p}$-adic fields, and their representations have been studied extensively in the past decades. In particular, the classification of irreducible representations and the character formulas have been achieved in [26,19,23], etc. The study uses in an essential way a convolution construction of the affine Hecke algebra using equivariant K-theory of Steinberg variety in [32].

Let G be a semisimple, simply-connected linear algebraic group over an algebraically closed field k. Consider the adjoint action of G on its Lie algebra $\mathfrak{g}$. The set of nilpotent elements in $\mathfrak{g}$ has a variety structure, called the nil-cone, denoted by $\mathcal{N}$. The G-action on $\mathcal{N}$ has finitely many orbits. The variety $\mathcal{N}$ is singular but admits a natural resolution of singularities, called the Springer resolution, as follows: Let $\mathcal{B}$ be the complete flag variety, parametrizing the set of Borel subalgebras of $\mathfrak{g}$. Let $\widetilde{\mathcal{N}}$ be its cotangent bundle $T^*\mathcal{B}$. Note that $T^*\mathcal{B}$ can be alternatively interpreted as the variety of pairs $(\mathfrak{b}, x)$, where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$, and x is a nilpotent element in $\mathfrak{b}$. There is a natural map $\widetilde{\mathcal{N}} \to \mathcal{N}$, sending each pair $(\mathfrak{b}, x)$ to x, which gives the resolution. For any $x \in \mathcal{N}$, the fiber of this map over x is called the *Springer fiber*, denoted by $\mathcal{B}_x$. Note that the Springer fibers are equi-dimensional, projective varieties, but in general not smooth. For any two different points in the same G-orbit of $\mathcal{N}$, the fibers are isomorphic. It is well-known that representations of the Weyl group of G can be constructed by looking at Borel–Moore homology of Springer fibers. This construction can be traced back to Springer, and later on was re-described and generalized by many others. In particular, Lusztig in [32] constructed an action of the affine Hecke algebra on equivariant K-theory of Springer fibers.

More precisely, in the constructions of Springer and Lusztig, they identified respectively the group algebra of the Weyl group and the affine Hecke algebra as the top Borel–Moore homology and respectively the equivariant K-theory of the Steinberg variety $Z := \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$, both endowed with convolution products. The essential property used about Borel–Moore homology and K-theory is that they both admit push-forwards for proper morphisms, and pull-backs for smooth morphisms. In fact, a functor from the category of smooth quasi-projective varieties to the category of commutative (graded) rings, that admits these two properties together with certain compatibility conditions, is