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Wave breaking in the Whitham equation

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ABSTRACT

We prove wave breaking — bounded solutions with unbounded derivatives — in the nonlinear nonlocal equation which combines the dispersion relation of water waves and a nonlinearity of the shallow water equations, provided that the slope of the initial datum is sufficiently negative, whereby we solve a Whitham's conjecture. We extend the result to equations of Korteweg–de Vries type for a range of fractional dispersion.

1. Introduction

As Whitham [22, pp. 457] emphasized, "the breaking phenomenon is one of the most intriguing long-standing problems of water wave theory." The *shallow water equations*:

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$$\begin{aligned} &\partial_t \eta + \partial_x((1+a\eta)u) = 0, \\ &\partial_t u + \partial_x \eta + au\partial_x u = 0, \end{aligned} \tag{1.1}$$

approximate the physical problem, and they explain *wave breaking*. That is, the solution remains bounded but its slope becomes unbounded in finite time. Here $t \in \mathbb{R}$ is proportional to elapsed time, and $x \in \mathbb{R}$ is the spatial variable in the primary direction of wave propagation; $\eta = \eta(x,t)$ is the fluid surface displacement from the depth $= 1$, $u = u(x,t)$ is the particle velocity at the rigid flat bottom, and $a > 0$ is the dimensionless nonlinearity parameter; see [16, Section 5.1.1.1], for instance, for details. Throughout, ∂ means partial differentiation. Note that the phase speed associated with the linear part of (1.1) is 1. In other words, the effects of dispersion do not live in (1.1). On the other hand, the phase speed for water waves is $c_{\rm ww}(\sqrt{b}\xi)$, after normalization of parameters, where

$$c^2_{\rm ww}(\xi) = \frac{\tanh \xi}{\xi}. \tag{1.2}$$

For relatively shallow water or, equivalently, relatively long waves satisfying $b \ll 1$, one may expand the right side of (1.2) and find that

$$c_{\rm ww}(\sqrt{b}\xi) = 1 - \frac{1}{6}b\xi^2 + O(b^2). \tag{1.3}$$

But the shallow water theory goes too far. It predicts that *all* solutions carrying an increase of elevation break. Yet observations have long since established that some waves do not break. Perhaps, the neglected dispersion effects inhabit breaking.

But including some dispersion effects (see (1.3)), the *Korteweg–de Vries (KdV) equation*:

$$\partial_t u + \left(1 + \frac{1}{6}b\partial_x^2\right)\partial_x u + \frac{3}{2}au\partial_x u = 0 \tag{1.4}$$

goes too far, and it predicts that *no* solutions break. As a matter of fact, the global-in-time well-posedness for (1.4) was established in [6], for instance, in $H^s(\mathbb{R})$ for $s \geqslant -3/4$.

To recapitulate, one necessitates some dispersion effects to satisfactorily explain wave breaking, but the dispersion of the KdV equation seems too strong for short waves. It is not surprising because the phase speed $= 1 - \frac{1}{6}b\xi^2$ associated with the linear part of (1.4) poorly approximates that for water waves when $b \gg 1$; see (1.3).

Whitham therefore noted that “it is intriguing to know what kind of simpler mathematical equation (than the governing equations of the water wave problem) could include” the breaking effects, and he [21] (see also [22, pp. 477]) put forward

$$\partial_t u + \int_{-\infty}^{\infty} K(x-y)\partial_y u(y,t)\, dy + \frac{3}{2}au\partial_x u = 0, \tag{1.5}$$