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Contact measures in isotropic spaces

Gil Solanes [1]

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia, Spain

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ABSTRACT

We revisit the contact measures introduced by Firey, and further developed by Schneider and Teufel, from the perspective of the theory of valuations on manifolds. This reveals a link between the kinematic formulas for area measures studied by Wannerer and the integral geometry of curved isotropic spaces. As an application we find explicitly the kinematic formula for the surface area measure in Hermitian space.

1. Introduction

The principal kinematic formula of Blaschke, Santaló and Chern states

$$\int_{\overline{SO(n)}} \chi(A \cap gB)dg = \omega_n^{-1} \sum_{i=0}^{n} \binom{n}{i}^{-1} \omega_i \omega_{n-i} \mu_i(A) \mu_{n-i}(B), \tag{1}$$

where χ is the Euler characteristic, and $A, B \subset \mathbb{R}^n$ are sufficiently nice compact subsets of $\mathbb{R}^n$ (e.g. convex bodies or smooth submanifolds). The integral over the group of rigid

E-mail address: solanes@mat.uab.cat.
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motions $\overline{SO(n)} = SO(n) \ltimes \mathbb{R}^n$ is performed with respect to the Haar measure dg, suitably normalized. On the right hand side, the constants ω_i denote the volume of the unit i-dimensional ball, and the functionals μ_i are the so-called *intrinsic volumes* (also known as quermassintegrals or Lipschitz–Killing curvature integrals). On the space of convex bodies $\mathcal{K}^n$, intrinsic volumes are *valuations*: a (real-valued) valuation is a functional $\varphi : \mathcal{K}^n \to \mathbb{R}$ such that

$$\varphi(K \cup L) = \varphi(K) + \varphi(L) - \varphi(K \cap L),$$

whenever $K, L, K \cap L \in \mathcal{K}^n$. Hadwiger's characterization theorem states that the space of continuous $\overline{SO(n)}$-invariant valuations is spanned by $\mu_0, \dots, \mu_n$. In particular, this fundamental result yields a simple proof of the principal kinematic formula for convex bodies.

Recent results by S. Alesker (cf. e.g. [4,6,7]) have allowed important progress in the theory of valuations. This includes, for instance, classification results and kinematic formulas for tensor-valued valuations (cf. e.g. [19,31,32]) and also for valuations taking values on the space of convex bodies (cf. e.g. [1,2,30,33,38,39]).

Another line of research is the determination of kinematic formulas with respect to different groups. Indeed, Alesker has shown that characterization theorems in the style of Hadwiger's exist also when the group $SO(n)$ is replaced by any compact group H acting transitively on the unit sphere. More precisely it was proved in [5] that the space Val^H of continuous, translation-invariant and H-invariant valuations has finite dimension. The connected groups H satisfying the previous condition were classified in [23,35]. For some of them, namely $H = SO(n), U(n), SU(n), Sp(2)Sp(1), G_2, Spin(7)$, a basis of the space Val^H has been constructed (cf. [7,11,12,20]). For the rest of groups, namely $H = Sp(n), Sp(n)U(1), Sp(n)Sp(1), Spin(9)$, only the dimension of Val^H is known (cf. [13,22]). Let us mention that the case of non-compact isotropy groups is also interesting and has been studied (cf. [10,16,34]).

As in the classical case, the fact that Val^H has finite dimension for any compact group H acting transitively on the unit sphere implies the existence of kinematic formulas in the style of (1) with respect to H. The actual computation of these formulas is a difficult problem, which has been recently solved in all cases where a basis of Val^H is known [11–13,17,21]. This was possible thanks to an algebraic approach developed by Bernig and Fu [17] and based on the product of valuations discovered by Alesker [8].

More recently, Alesker developed a theory of valuations on smooth manifolds (cf. [9] and the references therein). By a previous result of Fu [25], it turns out that kinematic formulas, expressible in terms of such valuations, exist in any Riemannian isotropic space. Such a space is a Riemannian manifold M together with a group G acting on M by isometries, and such that the induced action on the sphere bundle SM is transitive. The precise statement is the following: given a basis $\varphi_1, \dots, \varphi_d$ of the space $\mathcal{V}(M)^G$ of G-invariant valuations on M (cf. Definition 2), there exists