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Chern–Gauss–Bonnet and Lefschetz duality from a currential point of view

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We use the mapping cone for the relative deRham cohomology of a manifold with boundary in order to show that the Chern–Gauss–Bonnet Theorem for oriented Riemannian vector bundles over such manifolds is a manifestation of Lefschetz Duality in any of the two embodiments of the latter. We explain how Thom isomorphism fits into this picture, complementing thus the classical results about Thom forms with compact support. When the rank is *odd*, we construct, by using secondary transgression forms introduced here, a new closed pair of forms on the disk bundle associated to a vector bundle, pair which is Lefschetz dual to the zero section.

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1. Introduction

The mapping cone construction is a standard tool in algebraic topology. The purpose of this note is to put it to good use in the context of relative deRham cohomology on a manifold with boundary. So far, the Dirichlet representation of relative deRham classes as forms which pull-back to zero over the boundary has been by far the favorite sister in the literature. Nevertheless, we contend that working with (closed) pairs of forms has certain advantages such as turning explicit several maps of interest in cohomology. The mapping cone construction for deRham cohomology, whose definition can be encountered for example in the classical book of Bott and Tu [3] has rarely been used in the context of differential topology. However, its use sheds new light and helps extend certain classical results like the celebrated Chern–Gauss–Bonnet Theorem on manifolds with boundary.

The fact that Chern Theorem [6] on such manifolds is indeed a manifestation of Lefschetz Duality might not come as a surprise. After all, its boundaryless counterpart is a combination of two facts: an explicit Poincaré Duality statement plus the Poincaré–Hopf Theorem. It might also be expected to get in fact two Chern–Gauss–Bonnet statements corresponding to the two classes of isomorphisms:

$$\text{relative cohomology} \longleftrightarrow \text{absolute homology}$$
$$\text{absolute cohomology} \longleftrightarrow \text{relative homology}$$

What we found rather surprising is that one can recast the Chern–Gauss–Bonnet Theorem on a manifold with boundary also as a *direct realization* of Thom isomorphism.

The closed pair (Pfaffian, transgression of the Pfaffian) plays the same role as the Thom form with compact support when one considers on one side of the Thom isomorphism the relative cohomology of the pair (disk bundle, spherical bundle) instead of the (isomorphically equivalent) cohomology with compact supports. This is the *even* rank picture. Now Thom isomorphism holds irrespective of the parity of the rank, provided the vector bundle is oriented. Therefore, when the rank is *odd* we produce a new closed pair of forms on the same manifold with boundary that fulfills the same property as the already described pair in the even case. In particular, this pair is Lefschetz dual to the zero section. This result could be viewed as an *odd rank* version of the Chern–Gauss–Bonnet Theorem for a manifold with boundary. In both cases, the pairs can be used to define *explicit* Thom forms with compact support in a straightforward manner.

Before we take a look at some details let us say a few words about the techniques used to prove these results. Following the work of Harvey and Lawson and their school [11,14], we presented in [7] a general transgression formula for vertical, tame Morse–Bott–Smale