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How to avoid a compact set ☆

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A first-order expansion of the $\mathbb{R}$-vector space structure on $\mathbb{R}$ does not define every compact subset of every $\mathbb{R}^n$ if and only if topological and Hausdorff dimension coincide on all closed definable sets. Equivalently, if $A \subseteq \mathbb{R}^k$ is closed and the Hausdorff dimension of A exceeds the topological dimension of A, then every compact subset of every $\mathbb{R}^n$ can be constructed from A using finitely many boolean operations, cartesian products, and linear operations. The same statement fails when Hausdorff dimension is replaced by packing dimension.

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1. Introduction

We study metric dimensions and general geometric tameness of definable sets in expansions of the ordered real additive group $(\mathbb{R}, <, +)$. This is an important part of the so called *tameness program* first outlined by Miller [24]. As we hope our results are of interest to logicians and metric geometers alike, we begin with an essentially logic free motivation for, and description of, the main results of this paper. We then describe some of the more technical results and their connections to notions from pure model theory for readers with background in logic.

For metric geometers

A **structure** on $\mathbb{R}$ is a sequence $\mathfrak{S} := (\mathfrak{S}_m)_{m=1}^{\infty}$ such that for each m:

(1) $\mathfrak{S}_m$ is a boolean algebra of subsets of $\mathbb{R}^m$.
(2) If $X \in \mathfrak{S}_m$ and $Y \in \mathfrak{S}_n$, then $X \times Y \in \mathfrak{S}_{m+n}$.
(3) If $1 \le i \le j \le m$, then $\{(x_1, \dots, x_m) \in \mathbb{R}^m \ : \ x_i = x_j\} \in \mathfrak{S}_m$.
(4) If $X \in \mathfrak{S}_{m+1}$, then the projection of X onto the first m coordinates is in $\mathfrak{S}_m$.

Let K be a subfield of $\mathbb{R}$. We say a structure that satisfies the following three conditions is a K**-structure**.

(5) If $X \in \mathfrak{S}_m$ and $T : \mathbb{R}^m \to \mathbb{R}^n$ is K-linear, then $T(X) \in \mathfrak{S}_n$.
(6) If $1 \le i \le j \le m$, then $\{(x_1, \dots, x_m) \in \mathbb{R}^m \ : \ x_i < x_j\} \in \mathfrak{S}_m$.
(7) the singleton $\{r\} \in \mathfrak{S}_1$ for every $r \in \mathbb{R}$.

We say that $X \subseteq \mathbb{R}^n$ **belongs** to $\mathfrak{S}$ (or $\mathfrak{S}$ **contains** X) if $X \in \mathfrak{S}_n$. We say $\mathfrak{S}$ **avoids** X if $X \notin \mathfrak{S}_n$.

Given $A \subseteq \mathbb{R}^n$, we let $\mathfrak{M}_K(A)$ be the smallest K-structure containing A. When we do not specify K, we mean that $K = \mathbb{R}$. In particular, $\mathfrak{M}(A) = \mathfrak{M}_{\mathbb{R}}(A)$. Note that $\mathfrak{M}_K(\emptyset) \subseteq \mathfrak{M}_K(A)$ for any A. It is easy to see that a solution set of a finite system of K-affine inequalities is in $\mathfrak{M}_K(\emptyset)$. In particular every interval and every cartesian product of intervals is in $\mathfrak{M}_K(\emptyset)$. Every polyhedron is in $\mathfrak{M}(\emptyset)$.

It is natural to ask the following naive question:

Given A, what can be said about $\mathfrak{M}(A)$?

This is an important question in model theory in general and there are many classical results that provide answers to special cases. We give some examples.

- If A is the solution set of a finite system of inequalities between affine functions then every element of $\mathfrak{M}(A)$ is a finite union of solution sets of affine inequalities. This