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A note on Willmore minimizing Klein bottles in Euclidean space

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ABSTRACT

We show that $\varphi \circ \tilde{\tau}_{3,1} : K \to \mathbb{R}^4 \times \{0\}^{n-4}$ is the unique minimizer among immersed Klein bottles in its conformal class, where $\varphi : \mathbb{S}^4 \to \mathbb{R}^4$ is a stereographic projection and $\tilde{\tau}_{3,1}$ is the bipolar surface of Lawson's $\tau_{3,1}$-surface [11]. We conjecture that $\varphi \circ \tilde{\tau}_{3,1}$ is the unique minimizer among immersed Klein bottles into $\mathbb{R}^n$, $n \geq 4$, whose existence the authors and P. Breuning proved in [2].

1. Introduction

Let M be a closed manifold of dimension two. The Willmore energy of an immersed surface $f : M \to \mathbb{R}^n$, $n \geq 3$, is defined by

$$\mathcal{W}(f) := \frac{1}{4} \int_M |H|^2 d\mu_g,$$

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where H is the mean curvature vector of the surface, $g := f^\sharp \delta_{\mathbb{R}^n}$ the induced metric on M and $d\mu_g$ the induced area measure.

Willmore [19] proved 1965 that $\mathcal{W}(f) \geq 4\pi$ holds for any closed surface in $\mathbb{R}^3$ with equality for round spheres. He also computed the Willmore energy of certain tori and found out that the minimum of the Willmore energy among these tori is attained by a stereographic projection of the Clifford torus (with energy $2\pi^2$). He conjectured that every orientable surface in $\mathbb{R}^3$ of genus greater than zero satisfies

$$\mathcal{W}(f) \geq 2\pi^2.$$

This *Willmore conjecture* was proved by Marques and Neves [13]. Before the proof of Marques and Neves appeared a lot of partial results were obtained concerning the Willmore conjecture, see [14] and the references therein.

For non-orientable surfaces the number of results concerning the Willmore energy are quite limited. Li and Yau proved in [12] that $W(f) \geq 6\pi$ for any immersed $\mathbb{R}P^2$ in $\mathbb{R}^n$, $n \geq 4$, with equality if and only if $f : \mathbb{R}P^2 \to \mathbb{R}^4$ is the Veronese embedding. As there always is a triple point for immersed $\mathbb{R}P^2$ in $\mathbb{R}^3$ [1] an inequality from [12] gives $\mathcal{W}(f) \geq 12\pi$ with equality for example for Boy's surface [9,3].

In [2], the authors proved together with P. Breuning that the infimum among all immersed Klein bottles in $\mathbb{R}^n$, $n \geq 4$ is attained by a smooth embedding. The value of this minimum is strictly less than 8π. In this paper, we prove that it is less or equal to $6\pi\,\mathrm{E}\left(\frac{2\sqrt{2}}{3}\right) \approx 6.682\pi$, where $\mathrm{E}(.)$ is the complete elliptic integral of second kind. Our first result is the following:

Theorem 1.1. *Consider the bipolar surface of Lawson's $\tau_{3,1}$ torus and denote it by $\tilde{\tau}_{3,1}$. It is known that the surface $\tilde{\tau}_{3,1}$ is a minimally embedded Klein bottle in $\mathbb{S}^4$ [10]. Let $\varphi : \mathbb{S}^4 \to \mathbb{R}^4$ be a stereographic projection. Then we have that $\varphi \circ \tilde{\tau}_{3,1} : K \to \mathbb{R}^4 \times \{0\}^{n-4}$, $n \geq 4$, is the minimizer of the Willmore energy in its conformal class, i.e. we have that*

$$\mathcal{W}(f) \geq 6\pi\,\mathrm{E}\left(\frac{2\sqrt{2}}{3}\right) \approx 6.682\pi \tag{1}$$

for every immersed Klein bottle $f : K \to \mathbb{R}^n$, $n \geq 4$, that is conformal to $\varphi \circ \tilde{\tau}_{3,1}$. Here, $\mathrm{E}(.)$ is the complete elliptic integral of second kind. Furthermore, equality in (1) for an immersed Klein bottle f conformal to $\varphi \circ \tilde{\tau}_{3,1}$ implies that f is the surface $\varphi \circ \tilde{\tau}_{3,1}$ up to conformal diffeomorphisms of $\mathbb{R}^n$.

In the proof, we use the conformal volume studied by Li and Yau [12] and a result by Jakobson, Nadirashvili and Polterovich [8] who found out that $\tilde{\tau}_{3,1}$ is embedded by first eigenfunctions of the Laplacian. El Soufi, Giacomini and Jazar [4] proved that the metric on $\tilde{\tau}_{3,1}$ is the only metric on a Klein bottle that is critical for the first eigenvalue. This implies