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Tensor valuations on lattice polytopes

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ABSTRACT

The Ehrhart polynomial and the reciprocity theorems by Ehrhart & Macdonald are extended to tensor valuations on lattice polytopes. A complete classification is established of tensor valuations of rank up to eight that are equivariant with respect to the special linear group over the integers and translation covariant. Every such valuation is a linear combination of the Ehrhart tensors which is shown to no longer hold true for rank nine.

1. Introduction and statement of results

Tensor valuations on convex bodies have attracted increasing attention in recent years (see, e.g., [8,24,27]). They were introduced by McMullen in [38] and Alesker subsequently obtained a complete classification of continuous and isometry equivariant tensor valuations on convex bodies (based on [3] but completed in [5]). Tensor valuations have found applications in different fields and subjects; in particular, in Stochastic Geometry and

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Imaging (see [27]). The aim of this article is to begin to develop the theory of tensor valuations on lattice polytopes.

Let $\mathcal{P}(\mathbb{Z}^n)$ denote the set of lattice polytopes in $\mathbb{R}^n$; that is, the set of convex polytopes with vertices in the integer lattice $\mathbb{Z}^n$. In general, a full-dimensional lattice in $\mathbb{R}^n$ is an image of $\mathbb{Z}^n$ by an invertible linear transformation and, therefore, all results can easily be translated to the general situation of polytopes with vertices in an arbitrary lattice. A function Z defined on $\mathcal{P}(\mathbb{Z}^n)$ with values in an abelian semigroup is a *valuation* if

$$\mathrm{Z}(P) + \mathrm{Z}(Q) = \mathrm{Z}(P \cup Q) + \mathrm{Z}(P \cap Q)$$

whenever $P, Q, P \cup Q, P \cap Q \in \mathcal{P}(\mathbb{Z}^n)$ and $\mathrm{Z}(\emptyset) = 0$.

For $P \subset \mathbb{R}^n$, the lattice point enumerator, $\mathrm{L}(P)$, is defined as

$$\mathrm{L}(P) = \sum_{x \in P \cap \mathbb{Z}^n} 1. \tag{1}$$

Hence, $\mathrm{L}(P)$ is the number of lattice points in P and $P \mapsto \mathrm{L}(P)$ is a valuation on $\mathcal{P}(\mathbb{Z}^n)$. A function Z defined on $\mathcal{P}(\mathbb{Z}^n)$ is $\mathrm{SL}_n(\mathbb{Z})$ *invariant* if $\mathrm{Z}(\phi P) = \mathrm{Z}(P)$ for all $\phi \in \mathrm{SL}_n(\mathbb{Z})$ and $P \in \mathcal{P}(\mathbb{Z}^n)$ where $\mathrm{SL}_n(\mathbb{Z})$ is the special linear group over the integers; that is, the group of transformations that can be described by $n \times n$ matrices of determinant 1 with integer coefficients. A function Z is *translation invariant* on $\mathcal{P}(\mathbb{Z}^n)$ if $\mathrm{Z}(P + y) = \mathrm{Z}(P)$ for all $y \in \mathbb{Z}^n$ and $P \in \mathcal{P}(\mathbb{Z}^n)$. It is *i-homogeneous* if $\mathrm{Z}(k\,P) = k^i \mathrm{Z}(P)$ for all $k \in \mathbb{N}$ and $P \in \mathcal{P}(\mathbb{Z}^n)$ where $\mathbb{N}$ is the set of non-negative integers.

A fundamental result on lattice polytopes by Ehrhart [15] introduces the so-called Ehrhart polynomial and was the beginning of what is now known as Ehrhart Theory (see [6,7]).

Theorem *(Ehrhart). There exist* $\mathrm{L}_i : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ *for* $i = 0, \dots, n$ *such that*

$$\mathrm{L}(kP) = \sum_{i=0}^{n} \mathrm{L}_i(P) k^i$$

for every $k \in \mathbb{N}$ *and* $P \in \mathcal{P}(\mathbb{Z}^n)$*. For each* i*, the functional* L_i *is an* $\mathrm{SL}_n(\mathbb{Z})$ *and translation invariant valuation that is homogeneous of degree* i*.*

Note that $\mathrm{L}_n(P)$ is the n-dimensional volume, $V_n(P)$, and $\mathrm{L}_0(P)$ the Euler characteristic of P, that is, $\mathrm{L}_0(P) = 1$ for $P \in \mathcal{P}(\mathbb{Z}^n)$ non-empty and $\mathrm{L}_0(\emptyset) = 0$. Also note that $\mathrm{L}_i(P) = 0$ for $P \in \mathcal{P}(\mathbb{Z}^n)$ with $\dim(P) < i$, where $\dim(P)$ is the dimension of the affine hull of P.

Extending the definition of the lattice point enumerator (1), for $P \in \mathcal{P}(\mathbb{Z}^n)$ and a non-negative integer r, we define the *discrete moment tensor of rank* r by

$$\mathrm{L}^r(P) = \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} x^r$$